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Existence and Uniqueness Theorems for Some White Noise Integral Equations.

Dongya Zou
Louisiana State University and Agricultural & Mechanical College

EXISTENCE AND UNIQUENESS THEOREMS FOR SOME WHITE NOISE INTEGRAL EQUATIONS

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by
Dongya Zou
B.S., Xiangtan University, 1982
M.S., Xiangtan University, 1985
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UMI Number: 9637815

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

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Abstract

Let $(\mathcal{S})^*_\beta$, $0 \le \beta < 1$, be the Kondratiev-Streit spaces of generalized functions. Let $f : [0,T] \times (\mathcal{S})^*_\beta \to (\mathcal{S})^*_\beta$, be weakly measurable, and satisfy a growth condition and a Lipschitz condition. Let $\theta : [0,T] \to (\mathcal{S})^*_\beta$, be weakly measurable and satisfy a growth condition. Then it is shown that the white noise integral equation $X_t = \theta_t + \int_0^t f(s, X_s)ds, 0 \le t \le T$, has a unique solution in $(\mathcal{S})^*_\beta$, where the integral is a white noise integral in the Pettis or Bochner sense. This result is extended to $\mathcal{M}^*$, the Meyer-Yan distribution space.

Some special equations are also solved explicitly. For $F \in L^2(\mathbf{R}^+)$, let $A_s = \int_{-\infty}^s F(s-u)\partial_u du$, $E_s = \exp(A_s)$, and A_s^*, E_s^* be their duals, respectively. The equation $X_t = \theta_t + \int_0^t A_s^* X_s ds, t \in [0,T]$, is solved in $(\mathcal{S})^*$ or (L^2), and the equation $X_t = \theta_t + \int_0^t E_s^* X_s ds, t \in [0,T]$, is solved in $\mathcal{M}^*$, where θ is as above. Moreover, under certain conditions on $\theta, \Phi : [0,T] \to (\mathcal{S})^*$ and $\sigma : [0,T]^2 \to \mathbf{R}$, the Volterra equation $X_t = \theta_t + \int_0^t \sigma(t,s)\Phi_s \diamond X_s ds, t \in [0,T]$, is also solved, and its solution is in $\mathcal{M}^*$, $(\mathcal{S})^*_\beta$, or (L^2), depending on the growth conditions for θ and Φ. Finally, for a suitable deterministic function f, the white noise partial differential equation $\frac{\partial u}{\partial t} = \Delta u + :e^{\dot{B}_x}: \diamond u, u(0,x) = f(x), x \in \mathbf{R}^n, t \in [0,\infty)$, is solved in $\mathcal{M}^*$.

Introduction

Let $B(t)$ be a Brownian motion. The Itô theory of stochastic integration deals with the following stochastic integral and stochastic integral equation:

$$I(f) = \int_a^b f(t)\, dB(t), \tag{1}$$

$$X_t = x + \int_a^t g(s, X_s)\, dB(s) + \int_a^t h(s, X_s)\, ds, \quad a \le t \le b. \tag{2}$$

In equation (1) the stochastic process f is nonanticipating and almost all of its sample paths are in $L^2[a, b]$. In equation (2), the initial condition x is measurable with respect to the σ-field $\mathcal{F}_a$ generated by all $B_s, s \le a$. And the coefficients g, h: $[a, b] \times \mathbf{R} \to \mathbf{R}$, are assumed to satisfy the Lipschitz and growth conditions so that equation (2) has a unique continuous solution. The solution is a Markov process on [a, b] (see, e.g., [A]).

For anticipating stochastic process f, the integral in (1) can not be defined as an Itô integral. The white noise integral of the form

$$\int_a^b \partial_t^* f(t) dt, \tag{3}$$

which was studied in [HKPS, K2], is an extension of the Itô integral (1) to f that may not be nonanticipating. When (3) defines an L^2 random variable, the integral is called the Hitsuda-Skorokhod integral. If f is nonanticipating, and $f \in L^2([a, b] \times \mathcal{S}'(\mathbf{R}))$, then (3) and (1) coincide.

Assume that the initial condition x in (2) is not measurable with respect to $\mathcal{F}_a$. Since one usually uses the iteration method to show the existence of a solution of (2), then one may get an anticipating stochastic process $g(s, x)$. Therefore, for such initial condition x, we replace (2) by the following Hitsuda-Skorokhod type equation:

$$X_t = x + \int_a^t \partial_s^* g(s, X_s)\, ds + \int_a^t h(s, X_s)\, ds. \tag{4}$$

The solution is assumed to be an L^2 random variable. Kuo [K2] has studied (4) and obtained its L^2 solution for special functions g and h, that is, when $g(s,x) = g_1(s)x$, and $h(s,x) = h_1(s)x + h_2(s)$, or $g(s,x) = g_2(s)B(b)x$, and $h = 0$, where g_1, h_1 and g_2 are deterministic functions, and h_2 is a stochastic process. For details see Theorems 13.33 and 13.34 in [K2]. In general, it is very difficult to get an L^2 solution of (4). Thus we look for a solution in a bigger space than L^2, i.e., a generalized stochastic process in $(\mathcal{S})^*_\beta, \beta \in [0,1)$, the Kondratiev-Streit distribution spaces. Reasonable conditions on g and h have been given in [K1], and since then, Kuo has improved these results in [K2].

In fact, since we look for a solution which is a generalized stochastic process, we can combine the two integrals in equation (4) together. Moreover, we can replace the initial condition x by a more general function θ_t, and take $a = 0$ for simplicity. Thus we can study the following equation:

$$X_t = \theta_t + \int_0^t f(s, X_s)\, ds, \quad 0 \le t \le T. \tag{5}$$

Here the function θ is from $[0,T]$ into $(\mathcal{S})^*_\beta$, the function f is from $[0,T] \times (\mathcal{S})^*_\beta$ into $(\mathcal{S})^*_\beta$, and the integral is a white noise integral. This type of equation is quite general for applications. In this dissertation, we give more precise conditions on f than those in [K1] with the results suitable for $(\mathcal{S})^*_\beta$, and find conditions on θ so that equation (5) has a unique solution in $(\mathcal{S})^*_\beta$. When θ is a fixed element in $(\mathcal{S})^*_\beta$, equation (5) was also studied by Kuo [K2] under equivalent conditions on f. We will also extend our discussion to a wider space $\mathcal{M}^*$, the Meyer-Yan distribution space, since many equations have no solutions in $(\mathcal{S})^*_\beta$.

In this dissertation we will use a fundamental tool in white noise analysis, i.e. the S-transform. We not only use it to prove the existence and uniqueness theorems, but also to obtain explicit solutions for some particular equations.

For $F \in L^2(\mathbb{R}^+)$, define $A_s = \int_{-\infty}^s F(s-u)\partial_u du$, and $E_s = \exp(A_s)$. And let

A_s^* and E_s^* be their dual operators, respectively. These operators were studied by Oosawa et. al. [OTK] and Hida [H]. It turns out that A_s^* is an integral kernel operator studied in [HKPS, K2]. Using S-transform, we can solve the equation

$$X_t = \theta_t + \int_0^t A_s^* X_s ds, \quad 0 \leq t \leq T,$$

in (L^2) or $(\mathcal{S})^*$, and the equation

$$X_t = \theta_t + \int_0^t E_s^* X_s ds, \quad 0 \leq t \leq T,$$

in $\mathcal{M}^*$.

We also consider the following white noise Volterra type equation:

$$X_t = \theta_t + \int_0^t \sigma(t,s)\Phi_s \diamond X_s ds, \quad 0 \leq t \leq T,$$

where σ is a deterministic function, and $\theta, \Phi : [0,T] \to (\mathcal{S})_\beta^*$ are generalized functions. Under certain conditions on σ, θ, and Φ, we can construct its solution in $\mathcal{M}^*$. When $\Phi = \dot{B}_s$, then the solution is in $(\mathcal{S})^*$. Thus our result is an extension of the case studied by Øksendal and Zhang [ØZ] for a bounded function σ.

Finally we study the following white noise partial differential equation:

$$\frac{\partial u}{\partial t} = \Delta u + :e^{\dot{B}_x}: \diamond u, \quad u(0,x) = f(x), \quad x \in \mathbf{R}^n, \quad t \in [0,\infty).$$

We show that its solution belongs to $\mathcal{M}^*$. A similar equation has been studied by Holden et. al. [HLØUZ] by using the functional process approach.

Convention. Throughout this dissertation the phrase "null set" refers to a subset of $[0,T]$ in $\mathbf{R}$ of Lebesgue measure 0.

Chapter 1. White Noise Integration in $(\mathcal{S})^*_\beta$

1.1. The Kondratiev-Streit Spaces

Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space of real-valued rapidly decreasing functions on $\mathbf{R}$. Its dual space $\mathcal{S}'(\mathbf{R})$ consists of the tempered distributions. Then we have the following Gel'fand triple:

$$\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R}).$$

The Hermite function of degree $n, n \geq 0$, is defined by

$$e_n = \frac{(-1)^n}{\sqrt{n!2^n\sqrt{\pi}}} e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

It is obvious that $e_n \in \mathcal{S}(\mathbf{R})$ for $n \geq 0$, and the set $\{e_n; n \geq 0\}$ forms an orthonormal basis for $L^2(\mathbf{R})$.

Consider the operator

$$A = -\frac{d^2}{dx^2} + x^2 + 1,$$

which is densely defined on $L^2(\mathbf{R})$. Then e_n is an eigenfunction of A, with eigenvalue $2n+2$, i.e. $Ae_n = (2n+2)e_n$, for $n \geq 0$.

For each $p > \frac{1}{2}$, A^{-p} is a Hilbert-Schmidt operator of $L^2(\mathbf{R})$ and

$$\|A^{-p}\|^2_{HS} = \sum_{n=0}^{\infty} (2n+2)^{-2p}. \tag{1.1.1}$$

Define a family of norms on $L^2(\mathbf{R})$ by

$$|f|_p = |A^p f|_0 = \sqrt{\sum_{n=0}^{\infty} (2n+2)^{2p} \langle f, e_n \rangle_0^2}, \quad p \in \mathbf{R}.$$

Here $\langle \cdot, \cdot \rangle_0$ is the inner product in $L^2(\mathbf{R})$, and $|\cdot|_0$ the norm of $L^2(\mathbf{R})$. Denote

$$\mathcal{S}_p(\mathbf{R}) = \{f \in L^2(\mathbf{R}) : |f|_p < \infty\}, \quad p \geq 0.$$

Then $\mathcal{S}_p(\mathbf{R})$ is a Hilbert space, and

$$\mathcal{S}_q(\mathbf{R}) \subset \mathcal{S}_p(\mathbf{R}), \quad \forall q > p \geq 0.$$

Moreover, the Schwartz space $\mathcal{S}(\mathbf{R})$ is the projective limit of the family of the Hilbert spaces $\{\mathcal{S}_p(\mathbf{R}); p \geq 0\}$, i.e.

$$\mathcal{S}(\mathbf{R}) = \bigcap_{p\geq 0} \mathcal{S}_p(\mathbf{R}).$$

Let $\mathcal{S}_{-p}(\mathbf{R})$ be the completion of $L^2(\mathbf{R})$ with respect to $|\cdot|_{-p}$. It is easily seen that $\mathcal{S}_{-p}(\mathbf{R}) = \mathcal{S}_p'(\mathbf{R})$ is the dual space of $\mathcal{S}_p(\mathbf{R})$ for each $p \geq 0$, and $\mathcal{S}'(\mathbf{R})$ is the inductive limit of $\{\mathcal{S}_p'(\mathbf{R}) : p \geq 0\}$, i.e.

$$\mathcal{S}'(\mathbf{R}) = \bigcup_{p\geq 0} \mathcal{S}_p'(\mathbf{R}).$$

By Minlos' Theorem, there exists a standard Gaussian measure μ on $\mathcal{S}'(\mathbf{R})$, such that $\int_{\mathcal{S}'(\mathbf{R})} e^{i\langle x,\xi\rangle}\mu(dx) = e^{-\frac{1}{2}|\xi|_0^2}$, for $\xi \in \mathcal{S}(\mathbf{R})$, where $\langle x, \xi\rangle$ is the natural pairing between $\mathcal{S}'(\mathbf{R})$ and $\mathcal{S}(\mathbf{R})$. Let $(L^2) = L_c^2(\mathcal{S}'(\mathbf{R}), \mu)$. Here S_c is the complexification of the space S. By the Wiener-Itô decomposition theorem, we have for any $\varphi \in (L^2)$

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n\rangle, \quad f_n \in \hat{L}_c^2(\mathbf{R}^n),$$

where $: x^{\otimes n} := \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}(2k-1)!!(-1)^k x^{\otimes(n-2k)} \hat{\otimes} Tr^{\otimes k}$, and the series converges in (L^2) norm. Here $\hat{S}$ means the symmetrization of the space S. Moreover the (L^2)-norm of φ is given by

$$\|\varphi\|_0 = \sqrt{\sum_{n=0}^{\infty} n!|f_n|_0^2},$$

where $|\cdot|_0$ denotes the L^2-norm on $\hat{L}_c^2(\mathbf{R}^n)$. Define for $\beta \in [0, 1)$, $p \geq 0$

$$\begin{aligned} \|\varphi\|_{p,\beta} &= \sqrt{\sum_{n=0}^{\infty} (n!)^{1+\beta}|f_n|_p^2}, \\ \|\Phi\|_{-p,-\beta} &= \sqrt{\sum_{n=0}^{\infty} (n!)^{1-\beta}|f_n|_{-p}^2}, \end{aligned}$$

where $|f_n|_p = |(A^p)^{\otimes n} f_n|_0$, and $|f_n|_{-p} = |(A^{-p})^{\otimes n} f_n|_0$. Let

$$(\mathcal{S})_{p,\beta} = \{\varphi \in (L^2) : \|\varphi\|_{p,\beta} < \infty\}.$$

The space $(\mathcal{S})_{-p,-\beta}$ is defined to be the completion of (L^2) with respect to the norm $\|\cdot\|_{-p,-\beta}$. It is easily seen that $(\mathcal{S})_{-p,-\beta} = (\mathcal{S})^*_{p,\beta}$. For any fixed $\beta \in [0,1)$, we have

$$(\mathcal{S})_{q,\beta} \subset (\mathcal{S})_{p,\beta}, \quad \forall q > p \geq 0.$$

Let $(\mathcal{S})_\beta$ be the projective limit of the family of Hilbert spaces $\{(\mathcal{S})_{p,\beta}; p \geq 0\}$, i.e.

$$(\mathcal{S})_\beta = \bigcap\nolimits_{p\geq 0} (\mathcal{S})_{p,\beta}.$$

Its dual space $(\mathcal{S})^*_\beta$ is given by

$$(\mathcal{S})^*_\beta = \bigcup\nolimits_{p\geq 0} (\mathcal{S})^*_{p,\beta}.$$

When $\beta = 0$, we have $(\mathcal{S})_0 = (\mathcal{S})$ and $(\mathcal{S})^*_0 = (\mathcal{S})^*$, the Hida spaces. Now we have the following embedding chain of spaces:

$$(\mathcal{S})_\beta \subset (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* \subset (\mathcal{S})^*_\beta, \quad 0 \leq \beta < 1.$$

The S-transform will play a very important role in our study. For $\xi \in \mathcal{S}_c(\mathbf{R})$, we define the renormalization $:e^{\langle\cdot,\xi\rangle}:$ by setting

$$:e^{\langle\cdot,\xi\rangle}: := \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \frac{\xi^{\otimes n}}{n!}\rangle. \tag{1.1.2}$$

The following proposition can be found in [K2] (see Theorem 5.7 there).

PROPOSITION 1.1.1 *If $\xi \in \mathcal{S}_c(\mathbf{R})$, then $:e^{\langle\cdot,\xi\rangle}: \in (\mathcal{S})_\beta$, for any $\beta \in [0,1)$. Moreover the following estimate holds:*

$$\|:e^{\langle\cdot,\xi\rangle}:\|_{p,\beta} \leq 2^{\frac{\beta}{2}} \exp[(1-\beta)2^{\frac{2\beta-1}{1-\beta}} |\xi|_p^{\frac{2}{1-\beta}}], \quad \forall p \geq 0, \quad \beta \in [0,1).$$

Therefore for any $\Phi \in (\mathcal{S})^*_\beta$, we can define its S-transform by

$$S\Phi(\xi) = \ll \Phi, :e^{\langle \cdot, \xi \rangle}: \gg, \quad \xi \in \mathcal{S}_c(\mathbf{R}), \tag{1.1.3}$$

where $\ll \cdot, \cdot \gg$ is the natural pairing between $(\mathcal{S})^*_\beta$ and $(\mathcal{S})_\beta$. For each $\varphi \in (\mathcal{S})_\beta$ and $\Phi \in (\mathcal{S})^*_\beta$, with the following Wiener-Itô decompositions (see [K2]):

$$\varphi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n \rangle, \quad f_n \in \hat{\mathcal{S}}_c(\mathbf{R}^n), \tag{1.1.4}$$

$$\Phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F_n \rangle, \quad F_n \in \hat{\mathcal{S}}'_c(\mathbf{R}^n), \tag{1.1.5}$$

the following formula holds:

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n! < F_n, f_n > . \tag{1.1.6}$$

Combining (1.1.2), (1.1.3) and (1.1.6) we get

PROPOSITION 1.1.2 *For any $\Phi \in (\mathcal{S})^*_\beta$ with the Wiener-Itô decomposition (1.1.5), its S-transform is given by*

$$S\Phi(\xi) = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathcal{S}_c(\mathbf{R}).$$

The following three theorems can be found in [HKPS, K2].

THEOREM 1.1.3 *Let $\Phi \in (\mathcal{S})^*_\beta$. Then its S-transform $F(\xi) = S\Phi(\xi)$ satisfies the conditions:*

(a) For any $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, the function $F(\xi + z\eta), z \in \mathbf{C}$ is an entire function.

(b) There exist nonnegative constants K, a and p such that

$$|F(\xi)| \leq K \exp[a|\xi|_p^{\frac{2}{1-\beta}}], \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}).$$

Conversely, if a function $F(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$ satisfies the above two conditions, then there exists unique $\Phi \in (\mathcal{S})^*_\beta$, such that $F(\xi) = S\Phi(\xi)$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

Theorem 1.1.3 is called the characterization theorem in $(\mathcal{S})^*_\beta$. A function $F(\xi)$ is called a U-functional, if it satisfies the above conditions (a) and (b).

Given $\Phi, \Psi \in (\mathcal{S})^*_\beta$, $S\Phi$ and $S\Psi$ are both U-functionals. It is easy to check that their product is again a U-functional, therefore by the characterization theorem, there exists a unique element in $(\mathcal{S})^*_\beta$, denoted by $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\xi) = S\Phi(\xi) \cdot S\Psi(\xi)$.

Definition 1.1.4 *For $\Phi, \Psi \in (\mathcal{S})^*_\beta$, $\Phi \diamond \Psi$ is called the Wick product of Φ and Ψ.*

Theorem 1.1.5 *Let $F(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$, be a function on $\mathcal{S}_c(\mathbf{R})$ satisfying the following conditions:*

(a) For any $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, the function $F(\xi + z\eta), z \in \mathbf{C}$ is an entire function.

(b) There exist nonnegative constants K, a and p with $\|A^{-p}\|^2_{HS} < \frac{1}{2ae^2}$, such that

$$|F(\xi)| \le Ke^{a|\xi|^2_{-p}}, \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}).$$

Then there exists a unique $\varphi \in (L^2)$, such that $F(\xi) = S\varphi(\xi)$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$. Moreover

$$\|\varphi\|_0 \le K(1 - 2ae^2\|A^{-p}\|^2_{HS})^{-\frac{1}{2}}.$$

Theorem 1.1.6 *Let $\Phi_n \in (\mathcal{S})^*_\beta$ and $F_n(\xi) = S\Phi_n(\xi)$. Then Φ_n converges strongly in $(\mathcal{S})^*_\beta$ if and only if the following conditions are satisfied:*

(a) $\lim_{n\to\infty} F_n(\xi)$ exists for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(b) There exist nonnegative constants K, a and p, independent of n, such that

$$|F_n(\xi)| \le K \exp[a|\xi|_p^{\frac{2}{1-\beta}}], \quad \forall n \in \mathbf{N}, \xi \in \mathcal{S}_c(\mathbf{R}).$$

1.2. White Noise Integrals

In this section we study the white noise integrals in $(\mathcal{S})^*_\beta$ in the Bochner sense and Pettis sense.

Definition 1.2.1 *A function $\Phi : [0,T] \to (\mathcal{S})^*_\beta$ is Bochner integrable if it satisfies the following conditions:*

(1) Φ is weakly measurable.

(2) There exists a constant $p \geq 0$ such that $\Phi(s) \in (\mathcal{S})^*_{p,\beta}$ for almost all $s \in [0,T]$, and the function $\|\Phi(\cdot)\|_{-p,-\beta} \in L^1([0,T])$.

A function $\Phi : [0,T] \to (\mathcal{S})^*_\beta$ is Pettis integrable if it satisfies the following conditions:

(1) Φ is weakly measurable.

(2) $\ll \Phi(\cdot), \varphi \gg \in L^1([0,T])$ for each $\varphi \in (\mathcal{S})_\beta$.

(3) For each $t \in [0,T]$, there exists a generalized function in $(\mathcal{S})^*_\beta$, denoted by $\int_0^t \Phi(s)ds$, such that

$$\ll \int_0^t \Phi(s)ds, \varphi \gg = \int_0^t \ll \Phi(s), \varphi \gg ds, \quad \forall \varphi \in (\mathcal{S})_\beta.$$

A Bochner integrable function is obviously Pettis integrable.

Let $f : [0,T] \times (\mathcal{S})^*_\beta \to (\mathcal{S})^*_\beta$ satisfy the following conditions:

(A1) (Measurability Condition) The function $s \mapsto Sf(s,\Phi)(\xi)$ is measurable on [0, T] for each $\Phi \in (\mathcal{S})^*_\beta$ and $\xi \in \mathcal{S}_c(\mathbf{R})$.

(A2) (Growth Condition) There exist a Lebesgue integrable function $H(s), s \in [0,T]$, and nonnegative constants a and p, such that

$$|Sf(s,\Phi)(\xi)| \leq H(s)\exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + |S\Phi(\xi)|).$$

for any $\Phi \in (\mathcal{S})^*_\beta, \xi \in \mathcal{S}_c(\mathbf{R})$, and $a.e. s \in [0,T]$.

(A3) (Lipschitz Condition) There exist a function $L : [0,T] \times \mathcal{S}_c(\mathbf{R}) \to \mathbf{R}^+$, and nonnegative constants b and r, such that $\int_0^T L(s,\xi)ds \leq b(1 + |\xi|_r^{\frac{2}{1-\beta}})$, and

$$|Sf(s,\Phi)(\xi) - Sf(s,\Psi)(\xi)| \leq L(s,\xi)|S\Phi(\xi) - S\Psi(\xi)|$$

for any $\xi \in \mathcal{S}_c(\mathbf{R}), \Phi, \Psi \in (\mathcal{S})^*_\beta$, and $a.e. s \in [0,T]$.

Let $\Phi : [0,T] \to (\mathcal{S})^*_\beta$ satisfy the following conditions:

(C1)(Measurability Condition) The function $s \mapsto S\Phi_s(\xi)$ is measurable on [0, T] for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(C2)(Growth Condition) There exist nonnegative constants a_0, p_0, and a function $K_0(\cdot) \in L^1[0,T]$, with $K_0(\cdot)H(\cdot) \in L^1[0,T]$, and $K_0(\cdot)L(\cdot,\xi) \in L^1[0,T]$ (see conditions $(A2)$ and $(A3)$), such that

$$|S\Phi_s(\xi)| \leq K_0(s)\exp[a_0|\xi|_{p_0}^{\frac{2}{1-\beta}}], \tag{1.2.1}$$

for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$.

Denote $\mathcal{A}_f = \{\Phi : [0,T] \to (\mathcal{S})^*_\beta$ satisfying $(C1)$ and $(C2)\}$.

THEOREM 1.2.2 *Let $\Phi : [0,T] \to (\mathcal{S})^*_\beta$ satisfy conditions (C1) and (C2), with $H = 0$ and $L = 0$ in $(C2)$. Then*

(a) For any $q \geq p_0$ (see (C2)) such that $e^2(\frac{2a_0}{1-\beta})^{1-\beta}\|A^{-p}\|^2_{HS} < 1$,

$$\|\Phi_s\|_{-\beta,-q} \leq (1 - e^2(\frac{2a_0}{1-\beta})^{1-\beta}\|A^{-p}\|^2_{HS})K_0(s), \quad a.e. s \in [0,T].$$

(b) For each $t \in [0,T]$, the white noise integral $\int_0^t \Phi_s ds$ exists in the Bochner sense, whose S-transform is given by

$$S(\int_0^T \Phi_s ds)(\xi) = \int_0^T (S\Phi_s)(\xi)ds, \quad \xi \in \mathcal{S}_c(\mathbf{R}).$$

PROOF. Conclusion (a) follows from Theorem 8.2 [K2], and (b) follows from (a) and Definition 1.2.1. □

LEMMA 1.2.3 *If f satisfies conditions (A1-A3), then for each $p \geq 0$ and for a.e. $t \in [0,T]$, there exists $q \geq p$, with $e^2(\frac{2a_0}{1-\beta})^{1-\beta}\|A^{-p}\|^2_{HS} < 1$, such that, the function $f(t,\cdot)$ is continuous from $(\mathcal{S})^*_{p,\beta}$ to $(\mathcal{S})^*_{q,\beta}$.*

PROOF. Pick any $p_0 \geq 0$, any $\Phi \in (\mathcal{S})^*_{p_0,\beta}$, and any sequence $\{\Phi_n\} \subset (\mathcal{S})^*_{p_0,\beta}$ with Φ_n converges to Φ strongly. We need to show that there exists a Lebesgue null set N, and for each $t \notin N$, there exists $q \geq 0$ such that $f(t, \Phi_n)$ converges to $f(t, \Phi)$ strongly in $(\mathcal{S})^*_{q,\beta}$. Without loss of generality, we assume that $\|\Phi_n\|_{-p_0,-\beta} \leq 1 + \|\Phi\|_{-p_0,-\beta}$ for all n. Then by Proposition 1.1.1, we have for all n

$$\begin{aligned} |S\Phi_n(\xi)| &\leq 2\|\Phi_n\|_{-p_0,-\beta} \exp[c|\xi|_{p_0}^{\frac{2}{1-\beta}}] \\ &\leq 2(1 + \|\Phi\|_{-p_0,-\beta}) \exp[c|\xi|_{p_0}^{\frac{2}{1-\beta}}], \end{aligned}$$

where $c = (1-\beta)2^{\frac{2\beta-1}{1-\beta}}$ is a constant. By (A3), there exists a null set N_1, such that for each $t \notin N_1$, and each $\xi \in \mathcal{S}_c(\mathbf{R})$

$$Sf(t, \Phi_n)(\xi) \to Sf(t, \Phi)(\xi), n \to \infty. \tag{1.2.2}$$

Moreover by (A2), there exist a null set N_2, a nonnegative L^1-function $H(t)$, and nonnegative constants a, p such that for all n, for any $t \notin N_2$, and any $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} |Sf(t, \Phi_n)(\xi)| &\leq H(t) \exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + |S\Phi_n(\xi)|) \\ &\leq H(t) \exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + 2(1 + \|\Phi\|_{-p_0,-\beta}) \exp[c|\xi|_{p_0}^{\frac{2}{1-\beta}}]) \\ &\leq H_1(t) \exp[c_1|\xi|_{p_1}^{\frac{2}{1-\beta}}], \end{aligned}$$

where $H_1(t) = H(t)(3 + 2\|\Phi\|_{-p_0,-\beta})$ is Lebesgue integrable and $c_1 = c + a, p_1 = p_0 + p$. From this and (1.2.2), we have for all $t \notin N_3 = N_1 \cup N_2 \cup \{t : H(t) = \infty\}$, each $\xi \in \mathcal{S}_c(\mathbf{R})$

$$|Sf(t, \Phi)(\xi)| \leq H_1(t) \exp[c_1|\xi|_{p_1}^{\frac{2}{1-\beta}}]. \tag{1.2.3}$$

Therefore by Theorem 1.2.2, there exist a null set N_4, and a nonnegative number q, such that for all n and all $t \notin N_3 \cup N_4$

$$\|f(t, \Phi)\|_{-q,-\beta} \leq 2H_1(t), \quad \|f(t, \Phi_n)\|_{-q,-\beta} \leq 2H_1(t). \tag{1.2.4}$$

Therefore for all $t \notin N_3 \cup N_4$

$$\{f(t, \Phi), f(t, \Phi_n), \forall n\} \subset (\mathcal{S})^*_{q,\beta}. \tag{1.2.5}$$

For fixed $t \notin N_3 \cup N_4$, let $F_n(\xi) = Sf(t, \Phi_n)(\xi)$. Then, it is a Cauchy sequence satisfying the following uniform growth condition:

$$|F_n(\xi)| \leq H_1(t) \exp[c_1 |\xi|_{p_1}^{\frac{2}{1-\beta}}], \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}), n \geq 0.$$

By Theorem 1.1.6, there exists a unique $\Psi_t \in (\mathcal{S})^*_\beta$ such that $f(t, \Phi_n) \to \Psi_t$ in $(\mathcal{S})^*_\beta$ strongly. Therefore we have, for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $t \in [0, T]$

$$Sf(t, \Phi_n)(\xi) \to S\Psi_t(\xi), n \to \infty. \tag{1.2.6}$$

From (1.2.2), (1.2.6) and the injectivity of S-transform, we have $\Psi_t = f(t, \Phi)$ for almost all $t \in [0, T]$. This implies that $f(t, \Phi_n) \to f(t, \Phi)$ in $(\mathcal{S})^*_\beta$ strongly a.e. t. Thus $f(t, \Phi_n) \to f(t, \Phi)$ in $(\mathcal{S})^*_{q,\beta}$ strongly, a.e.t. by (1.2.5). □

LEMMA 1.2.4 *Let f satisfy conditions (A1-A3) and $\Phi \in \mathcal{A}_f$. Then there exists $q \geq 0$ such that the function $t \mapsto f(t, \Phi_t)$ from [0, T] into $(\mathcal{S})^*_{q,\beta}$ is Bochner integrable. Moreover, the generalized function $t \mapsto \int_0^t f(s, \Phi_s) ds$ is also in $\mathcal{A}_f$.*

PROOF. By combining the growth conditions for f and Φ we get for almost all $t \in [0, T]$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} |Sf(t, \Phi_t)(\xi)| &\leq H(t) \exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + |S\Phi_t(\xi)|) \\ &\leq H(t) \exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + K_0(t) \exp[a_0 |\xi|_{p_0}^{\frac{2}{1-\beta}}]) \\ &\leq H_2(t) \exp[a_2 |\xi|_{p_2}^{\frac{2}{1-\beta}}], \end{aligned}$$

where $H_2(t) = H(t) + H(t)K_0(t) \in L^1[0, T], a_2 = a + a_0$, and $p_2 = p + p_0$. Thus the generalized function $t \mapsto f(t, \Phi_t)$ satisfies condition (C2) with $H = 0$ and $L = 0$. We show next that it satisfies the measurability condition (C1). If $\Phi_t = \sum_{k=0}^n \Phi_k \chi_{B_k}(t)$, where the Φ_k's are elements in $(\mathcal{S})^*_\beta$ and the B_k's are pairwise disjoint Lebesgue sets in [0, T]. For $t \in B_k$, $f(t, \Phi_t) = f(t, \Phi_k)$, and so $f(t, \Phi_t) = \sum_{k=0}^n \chi_{B_k}(t) f(t, \Phi_k)$. For any $\xi \in \mathcal{S}_c(\mathbf{R}), Sf(t, \Phi_t)(\xi) = \sum_{k=0}^n \chi_{B_k}(t) Sf(t, \Phi_k)(\xi)$, which is a measurable function of t by condition $(A1)$ on f. For any $\Phi \in \mathcal{A}_f$, by Theorem 1.2.2, there exists $r \geq 0$ such that

$$\Phi_t \in (\mathcal{S})^*_{r,\beta} \quad a.e.t.$$

Thus by the measurability condition (C2) for Φ and a limiting argument we conclude that $\Phi_t, t \in [0,T]$, is weakly measurable. Therefore by Pettis' Theorem, $\Phi_t, t \in [0,T]$, is strongly measurable. Then there exists a sequence of simple functions $\Phi_t^{(n)}$ converging to Φ_t in $(\mathcal{S})^*_{r,\beta}, a.e.t.$ By Lemma 1.2.3, for $a.e.t.$ there exists $q \geq 0$, such that $f(t, \Phi_t^{(n)}) \to f(t, \Phi_t)(n \to \infty)$, in $(\mathcal{S})^*_{q,\beta}$. Therefore for any $\xi \in \mathcal{S}_c(\mathbf{R})$ we have $Sf(t, \Phi_t^{(n)})(\xi) \to Sf(t, \Phi_t)(\xi)(n \to \infty)$ a.e.t. But $Sf(t, \Phi_t^{(n)})(\xi), t \in [0,T]$ is measurable by the first part of the argument, therefore $Sf(t, \Phi_t)(\xi), t \in [0,T]$, is also measurable. This implies that the generalized function $t \mapsto f(t, \Phi_t)$ satisfies condition (C1). Now by Theorem 1.2.2, there exists $q \geq 0$, such that $f(t, \Phi_t) \in (\mathcal{S})^*_{q,\beta}, a.e.t$, and the white noise integral $\int_0^t f(s, \Phi_s)ds$ exists in the Bochner sense for each $t \in [0,T]$. This proves the first part of the lemma.

For any $\xi \in \mathcal{S}_c(\mathbf{R})$, the function

$$t \mapsto S(\int_0^t f(s, \Phi_s)ds)(\xi) = \int_0^t Sf(s, \Phi_s)(\xi)ds, \quad t \in [0,T]$$

is absolutely continuous, therefore measurable. Thus $t \mapsto \int_0^t f(s, \Phi_s)ds$ satisfies (C1). Moreover

$$\begin{aligned} &|S(\int_0^t f(s, \Phi_s)ds)(\xi)| = |\int_0^t Sf(s, \Phi_s)(\xi)ds| \\ \leq \ & (\int_0^t H_2(s)ds) \exp[a_2|\xi|_{p_2}^{\frac{2}{1-\beta}}] \leq (\int_0^T H_2(s)ds) \exp[a_2|\xi|_{p_2}^{\frac{2}{1-\beta}}]. \end{aligned}$$

Thus $t \mapsto \int_0^t f(s, \Phi_s)ds$ satisfies (C2) with $K_0(s) = \int_0^T H_2(s)ds =$ constant. Therefore $t \mapsto \int_0^t f(s, \Phi_s)ds$ is in $\mathcal{A}_f$, and Lemma 1.2.4 is proved. $\square$

LEMMA 1.2.5 *(Gronwall-Bellman inequality) Let m be a nonnegative measurable function on $[0,T]$ such that*

$$m(t) \leq h(t) + \int_0^t g(s)m(s)\,ds, \quad a.e. t \in [0,T],$$

where $h \in L^\infty([0,T])$, $g \geq 0$, $g \in L^1([0,T])$ and $gm \in L^1([0,T])$. Then

$$m(t) \leq h(t) + \int_0^t h(s)g(s)e^{\int_s^t g(u)\,du}\,ds \quad a.e. t \in [0,T].$$

PROOF. Define

$$z(t) = \int_0^t g(s)m(s)\,ds, \quad t \in [0,T].$$

Then $\dot{z}(t) = g(t)m(t)$ a.e. and it follows from the assumption that

$$\begin{aligned} \dot{z}(t) - g(t)z(t) &= g(t)(m(t) - z(t)) \\ &\leq g(t)h(t), \quad a.e. t \in [0,T]. \end{aligned}$$

By multiplying the integrator $\exp\left[-\int_0^t g(s)\,ds\right]$ to both sides of this inequality and then integrating, we obtain easily that, for almost all $t \in [0,T]$

$$z(t) \leq \int_0^t g(s)h(s)e^{\int_s^t g(u)\,du}\,ds.$$

The conclusion follows from this inequality and the assumption. □

1.3. White Noise Integral Equations

Consider the following white noise integral equation:

$$X_t = \theta_t + \int_0^t f(s, X_s)\,ds, \quad t \in [0,T], \tag{1.3.1}$$

where $\theta : [0,T] \to (\mathcal{S})_\beta^*$ and $f : [0,T] \times (\mathcal{S})_\beta^* \to (\mathcal{S})_\beta^*$ are generalized functions, and $\int_0^t f(s, X_s)\,ds$ is the white noise integral in the Bochner or Pettis sense.

DEFINITION 1.3.1 *A generalized function $X : [0,T] \to (\mathcal{S})_\beta^*$ is called a solution of equation (1.3.1) on $[0,T]$ if it satisfies the following conditions:*

(1) (Measurability Condition) X is weakly measurable.

(2) (Integrability Condition) The function $s \mapsto f(s, X_s), s \in [0,T]$ is Bochner or Pettis integrable.

(3) For any $\varphi \in (\mathcal{S})_\beta$, and a.e.$t \in [0,T]$, the following equality holds:

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \int_0^t \ll f(s, X_s), \varphi \gg ds.$$

Now we present our existence and uniqueness theorem for equation (1.3.1) in $(\mathcal{S})_\beta^*$.

THEOREM 1.3.2 *Let f satisfy conditions (A1)-(A3) and $\theta \in \mathcal{A}_f$. Then equation (1.3.1) has a unique solution $X : [0,T] \to (\mathcal{S})_\beta^*$, satisfying condition (C2).*

PROOF. To show uniqueness, let $\tilde{X} : [0,T] \to (\mathcal{S})_\beta^*$ be another solution of (1.3.1). Let $F_t = \mathcal{S}X_t$, $\tilde{F}_t = \mathcal{S}\tilde{X}_t$. Then, by (A3) we have for each $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} |F_t(\xi) - \tilde{F}_t(\xi)| &\leq \int_0^t |\mathcal{S}f(s, X_s) - \mathcal{S}f(s, \tilde{X}_s)|\, ds \\ &\leq \int_0^t L(s,\xi)|F_s(\xi) - \tilde{F}_s(\xi)|\, ds, \quad a.e.t \in [0,T]. \end{aligned}$$

Hence by Gronwall-Bellman ineqality, $F_t(\xi) = \tilde{F}_t(\xi)$ for almost all $t \in [0,T]$. That is, for each fixed $\xi \in \mathcal{S}_c(\mathbf{R})$, there exists a null set N_ξ in [0, T] such that $F_t(\xi) = \tilde{F}_t(\xi)$ for all $t \notin N_\xi$. Since $\mathcal{S}(\mathbf{R})$ is separable, there exists a countable sequence $\{\xi_n\}_{n\geq 1}$ dense in $\mathcal{S}(\mathbf{R})$. Let $N_0 = \cup_{n=1}^\infty N_{\xi_n}$. Then for each $t \notin N_0$ and $n \geq 1$, $F_t(\xi_n) = \tilde{F}_t(\xi_n)$. By a limiting argument we have $F_t(\xi) = \tilde{F}_t(\xi)$, for all $t \notin N_0$ and $\xi \in \mathcal{S}(\mathbf{R})$. This implies that $F_t(\xi) = \tilde{F}_t(\xi)$, for all $t \notin N_0$ and $\xi \in \mathcal{S}_c(\mathbf{R})$. By the injectivity of S-transform we get: $X_t = \tilde{X}_t$ for a.e.$t \in [0,T]$, thus showing the uniqueness.

In order to show the existence part, we apply the classical iteration method. For each $t \in [0,T]$, we define $X_t^{(0)} = \theta_t$. And for each $n = 1, 2, \cdots$, we define $X_t^{(n)}$

inductively:

$$X_t^{(n+1)} = \theta_t + \int_0^t f(s, X_s^{(n)})\, ds, \quad t \in [0,T]. \tag{1.3.2}$$

Since $\theta \in \mathcal{A}_f$, by Lemma 1.2.4 and induction, we see that for each $n \geq 0$, the white noise integral $\int_0^t f(s, X_s^{(n)})\, ds$, exists in the Bochner sense and $\{X^{(n)}\}_{n\geq 0} \subset \mathcal{A}_f$. Let $F_t^{(n)}(\xi) = SX_t^{(n)}(\xi)$, for $\xi \in \mathcal{S}_c(\mathbf{R})$. By combining the growth conditions for f and θ, we get a null set A_0, such that for all $s \notin A_0$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$|Sf(s,\theta_s)(\xi)| \leq K_1(s) \exp[a_1 |\xi|_{p_1}^{\frac{2}{1-\beta}}],$$

where $K_1 = H(1 + K_0) \in L^1[0,T]$, and a_1, p_1 are nonnegative constants. Thus we get for all $s \notin A_0$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} & |F_t^{(1)}(\xi) - F_t^{(0)}(\xi)| = |\int_0^t Sf(s,\theta_s)(\xi)\, ds| \\ \leq\ & \exp[a_1|\xi|_{p_1}^{\frac{2}{1-\beta}}] \int_0^T K_1(s)\, ds = c_1 \exp[a_1 |\xi|_{p_1}^{\frac{2}{1-\beta}}], \end{aligned}$$

where $c_1 = \int_0^T K_1(s) ds$ is a nonnegative constant. Note that c_1, a_1 and p_1 depend only on f and θ. From (A3) we get a null set A_1, such that for all $t \notin A_1$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} & |F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)| \\ \leq\ & \int_0^t L(s,\xi) |F_s^{(n)}(\xi) - F_s^{(n-1)}(\xi)| ds \\ \leq\ & \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} L(s_1,\xi) L(s_2,\xi) \cdots L(s_n,\xi) \\ & \cdot \sup_{s\notin A_0} |F_s^{(1)}(\xi) - F_s^{(0)}(\xi)| ds_1 ds_2 \cdots ds_n \\ \leq\ & \frac{1}{n!} (\int_0^t L(s,\xi)\, ds)^n c_1 \exp[a_1 |\xi|_{p_1}^{\frac{2}{1-\beta}}] \\ \leq\ & \frac{1}{n!} (b(1 + |\xi|_r^{\frac{2}{1-\beta}}))^n c_1 \exp[a_1|\xi|_{p_1}^{\frac{2}{1-\beta}}]. \end{aligned}$$

Hence we have for all $t \notin A_1$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} \sum_{n=0}^{\infty} |F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)| & \leq\ c_1 \exp[a_1 |\xi|_{p_1}^{\frac{2}{1-\beta}} + b(1 + |\xi|_r^{\frac{2}{1-\beta}})] \\ & \leq\ c_2 \exp[a_2 |\xi|_{p_2}^{\frac{2}{1-\beta}}], \end{aligned}$$

where $c_2 = c_1 e^b$, $a_2 = a_1 + b$ and $p_2 = p_1 + r$. This means that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, the series $\sum_{n=0}^{\infty}[F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)] + F_t^{(0)}$ converges uniformly for $t \notin A_1$. Denote the limit by $F(t)$. Then $F_t^{(n)}$ converges to $F(t)$ uniformly for $t \notin A_1$. On the other hand, by the growth condition for θ, there exists a null set A_2, such that, for all $t \notin A_1 \cup A_2$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$
\begin{aligned}
|F_t^{(n)}(\xi)| &= |F_t^{(0)}(\xi) + \sum_{m=0}^{n-1}[F_t^{(m+1)}(\xi) - F_t^{(m)}(\xi)]| \\
&\leq |F_t^{(0)}(\xi)| + \sum_{m=0}^{\infty}|F_t^{(m+1)}(\xi) - F_t^{(m)}(\xi)| \\
&\leq |S\theta_t(\xi)| + c_2 \exp[a_2|\xi|_{p_2}^{\frac{2}{1-\beta}}] \\
&\leq K_0(t)\exp[a_0|\xi|_p^{\frac{2}{1-\beta}}] + c_2\exp[a_2|\xi|_{p_2}^{\frac{2}{1-\beta}}] \\
&\leq H_1(t)\exp[a_3|\xi|_{p_3}^{\frac{2}{1-\beta}}],
\end{aligned}
$$

where $H_1 = K_0(t) + c_2$, $a_3 = a_2 + a_0 = a_1 + b + a_0$ and $p_3 = p_2 + p_0 = p_0 + r + p_1$. These constants and the function H_1 are independent of n. By letting $n \to \infty$, we have

$$
|F_t(\xi)| \leq H_1(t)\exp[a_3|\xi|_{p_3}^{\frac{2}{1-\beta}}]. \tag{1.3.3}
$$

We show next that $F_t(\cdot)$ is a U-functional for any fixed $t \notin A_1 \cup A_2$. For any fixed ξ and η in $\mathcal{S}_c(\mathbf{R})$, each function $F_t^{(n)}(z\xi + \eta), z \in \mathbf{C}$ is entire for $n \geq 0$. Moreover, it is easily seen that $F_t^{(n)}(z\xi + \eta)$ converges uniformly on any compact subset of $\mathbf{C}$ to $F_t(z\xi + \eta)$ (since $|z\xi + \eta|_p^{\frac{2}{1-\beta}} = |z|^{\frac{2}{1-\beta}}|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}}$). Therefore the function $F_t(z\xi + \eta), z \in \mathbf{C}$ is entire. This and inequality (1.3.3) imply that F_t is a U-functional. So by Theorem 1.1.3, there exists $X_t \in (\mathcal{S})^*_\beta$ such that $SX_t = F_t$. Pick and fix $t \notin A_1 \cup A_2$. Because $F_t^{(n)}(\xi)$ converges to $F_t(\xi)$ for each $\xi \in \mathcal{S}_c(\mathbf{R})$, and because $F_t^{(n)}$ and F_t satisfy uniform growth estimates for all $\xi \in \mathcal{S}_c(\mathbf{R})$, we can apply Theorem 1.1.6 to conclude that $X_t^{(n)}$ converges strongly to X_t. Since each $X^{(n)}$ is weakly measurable, so is X. Thus X satisfies the measurability condition in Definition 1.3.1. Furthermore, since $H_1 = K_0 + c_2$, $H_1 H$ is integrable. Combining this and (1.3.3) we see that X satisfies the growth condition $(C2)$. Thus $X \in \mathcal{A}_{\xi}$. This implies that $f(s, X_s)$ is Bochner integrable by Lemma 1.2.4, thus X satisfies the integrability condition in Definition 1.3.1. In particular, $Sf(s, X_s), s \in [0, T]$ is integrable on [0, T] . By taking the S-transform of both sides of equation (1.3.2) we get

$$F_t^{(n+1)}(\xi) = F_t^{(0)}(\xi) + \int_0^t Sf(s, X_s^{(n)})(\xi) ds. \tag{1.3.4}$$

Now by the Lipschitz condition (A3)

$$|Sf(s, X_s^{(n)})(\xi) - Sf(s, X_s)(\xi)| \leq L(s, \xi)|SX_s^{(n)}(\xi) - SX_s(\xi)|,$$

for all ξ and a.e.s. Thus $Sf(s, X_s^{(n)})(\xi)$ converges to $Sf(s, X_s)(\xi)$, for all ξ and a.e.s. Moreover, for each $t \in [0, T]$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\begin{aligned} & \left|\int_0^t Sf(s, X_s^{(n)})(\xi) ds - \int_0^t Sf(s, X_s)(\xi) ds\right| \\ \leq & \int_0^t |Sf(s, X_s^{(n)})(\xi) - Sf(s, X_s)(\xi)| ds \\ \leq & \int_0^t L(s, \xi)|SX_s^{(n)}(\xi) - SX_s(\xi)| ds \\ \leq & \int_0^T L(s, \xi) ds \sup_{s \notin A_1} |F_s^{(n)} - F_s| \\ \rightarrow & \ 0, n \rightarrow \infty, \end{aligned}$$

since $F_s^{(n)}$ converges to F_s uniformly for $s \notin A_1$. Thus for all $t \in [0, T]$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$

$$\int_0^t Sf(s, X_s^{(n)})(\xi)\, ds \rightarrow \int_0^t Sf(s, X_s)(\xi)\, ds, n \rightarrow \infty.$$

Therefore, by letting $n \rightarrow \infty$ in (1.3.4) we get for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and $t \notin A_1$

$$SX_t(\xi) = S\theta_t(\xi) + \int_0^t Sf(s, X_s)(\xi)\, ds. \tag{1.3.5}$$

Since the white noise integral $\int_0^t f(s, X_s) ds$ exists in the Pettis sense, we have

$$\ll \int_0^t f(s, X_s) ds, \varphi \gg = \int_0^t \ll f(s, X_s), \varphi \gg ds, \quad \forall \varphi \in (\mathcal{S})_\beta. \tag{1.3.6}$$

In particular

$$S(\int_0^t f(s,X_s)ds)(\xi) = \int_0^t Sf(s,X_s)(\xi)ds, \quad \forall\xi \in \mathcal{S}_c(\mathbf{R}). \tag{1.3.7}$$

Combining (1.3.5) with (1.3.7) we have, for all ξ and *a.e.t*

$$SX_t = S[\theta_t + \int_0^t f(s,X_s)ds]. \tag{1.3.8}$$

Since S-transform is injective, then (1.3.8) implies that for *a.e.t*

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \ll \int_0^t f(s,X_s)ds, \varphi \gg, \quad \forall\varphi \in (\mathcal{S})_\beta. \tag{1.3.9}$$

Combining (1.3.6) with (1.3.9) we get for a.e.t

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \int_0^t \ll f(s,X_s), \varphi \gg ds, \quad \forall\varphi \in (\mathcal{S})_\beta.$$

This implies that, X satisfies condition (3) in Definition 1.3.1. Thus X is a solution of (1.3.1). This proves the existence part of the theorem. Thus the theorem is proved. □

REMARKS (1) Compare this theorem with the existence and uniqueness theorem in [K2] (see Theorem 13.43). We have a different initial condition, therefore our solution satisfies the growth condition (C2) instead of being essential bounded as in Theorem 13.43. Moreover, following a similar argument as in Theorem 2.3.2, Chapter 2, we can replace our growth condition by that in Theorem 13.43 to get a unique solution X, but the function $s \mapsto f(s,X_s)$ is only Pettis integrable on [0, T].

(2) Let $f, \theta, X^{(n)}$ and X be as in Theorem 1.3.2. Then, it follows from the proof of the theorem, the growth conditions for f, θ, and Theorem 1.1.6 that, the integral $\int_0^t f(s,X_s)ds$ is a white noise integral in the Bochner sense. Moreover, taking $n \to \infty$ in equation (1.3.2), we have: $X_t^{(n)} \to X_t$, and $\int_0^t f(s,X_s^{(n)})ds \to \int_0^t f(s,X_s)ds$ strongly in $(\mathcal{S})_\beta^*$.

1.4. Examples

The examples in this section can all be solved in $(\mathcal{S})^*_\beta, \beta \in [0,1)$, except the last one, for which we will come back and look for its unique solution in a wider space, i.e., the Meyer-Yan distribution space $\mathcal{M}^*$.

Let δ_a be the Dirac delta function at a. The following fact will be used in Example 1.4.2, and several times later in this paper.

FACT *For any $\epsilon > 0$, there exists $p \geq 0$ such that $|\delta_s|_{-p} \leq \epsilon$, for any s.*

PROOF. By definition of the norm $|\cdot|_{-p}$, we have

$$\begin{aligned}
|\delta_s|_{-p}^2 &= |A^{-p}\delta_s|_0^2 \\
&= \sum_{n=0}^{\infty}(2n+2)^{-2p}\langle\delta_s, e_n\rangle^2 \\
&= \sum_{n=0}^{\infty}(2n+2)^{-2p}e_n^2(s) \\
&\leq \sum_{n=0}^{\infty}(2n+2)^{-2p}\|e_n\|_\infty^2 \\
&= \sum_{n=0}^{\infty}(2n+2)^{-2p}O(n^{-\frac{1}{12}}).
\end{aligned}$$

The last equality is from formula (21.3.3) in [HP]. For any $\epsilon > 0$, choose $p = p(\epsilon)$ such that $\sum_{n=0}^{\infty}(2n+2)^{-2p}n^{-\frac{1}{12}} < \epsilon^2$. Thus $|\delta_s|_{-p} < \epsilon$ for any s, and the fact is proved. □

EXAMPLE 1.4.1 Let $B(t) = \langle\cdot, 1_{[0,t)}\rangle$ be a Brownian motion. Then its k+1-th derivative is given by:

$$B^{(k+1)}(t) = (-1)^k\langle\cdot, \delta_t^{(k)}\rangle, \quad (k \geq 0). \tag{1.4.1}$$

Consider for $t \in [0,T]$ and $k \geq 0$, the following equation

$$X_t = 1 + B^{(k)}(t) + \int_0^t B^{(k+1)}(s) \diamond X_s\, ds, \quad t \in [0,T], \tag{1.4.2}$$

where $\diamond$ is the Wick product defined in Definition 1.1.4. Denote $\theta_t = B^{(k)}(t)$, and $f(s,\Phi) = B^{(k+1)}(s)\diamond\Phi$, then f satisfies conditions (A1-A3) for $\beta = 0$, and $\theta \in \mathcal{A}_f$.

Therefore there exists a unique solution of (1.4.2) by Theorem 1.3.2. Moreover, we claim that the solution is given by

$$X_t = [1 + B^{(k)}(0)]\diamond : e^{B^{(k)}(t) - B^{(k)}(0)} :\in (\mathcal{S})^*. \tag{1.4.3}$$

To check this claim, we consider $k = 0$ and $k > 0$ separately.

Case 1. $k = 0$. In this case, (1.4.2) becomes

$$X_t = 1 + B(t) + \int_0^t \dot{B}(s) \diamond X_s\, ds, \quad t \in [0, T]. \tag{1.4.4}$$

By taking $\mathcal{S}$-transform on both sides and denoting $\mathcal{S}X_t(\xi) \equiv F_t(\xi)$, we get

$$\begin{aligned} F_t(\xi) &= 1 + \int_0^t \xi(s)\, ds + \int_0^t \xi(s) F_s(\xi)\, ds \\ &= 1 + \int_0^t \xi(s)[1 + F_s(\xi)]\, ds. \end{aligned}$$

Its solution is given by

$$F_t(\xi) = e^{\langle \xi, 1_{[0,t)} \rangle},$$

therefore the solution of (1.4.4) is given by

$$X_t =: e^{\langle \cdot, 1_{[0,t)} \rangle} :=: e^{B(t)} : . \tag{1.4.5}$$

Since $B(0) = 0$, we have showed that (1.4.3) is the solution of (1.4.2) when $k = 0$.

Case 2. $k > 0$. In this case, we can rewrite (1.4.2) as the following

$$X_t = 1 + (-1)^{k-1} \langle \cdot, \delta_t^{(k-1)} \rangle + \int_0^t (-1)^k \langle \cdot, \delta_s^{(k)} \rangle \diamond X_s\, ds. \tag{1.4.6}$$

Taking S-transform on both sides we get

$$\begin{aligned} F_t(\xi) &= 1 + (-1)^{k-1} \langle \xi, \delta_t^{(k-1)} \rangle + (-1)^k \int_0^t \langle \xi, \delta_s^{(k)} \rangle F_s(\xi)\, ds \\ &= 1 + \langle \delta_t, \xi^{(k-1)} \rangle + \int_0^t \langle \delta_s, \xi^{(k)} \rangle F_s(\xi)\, ds \\ &= 1 + \xi^{(k-1)}(t) + \int_0^t \xi^{(k)}(s) F_s(\xi)\, ds. \end{aligned}$$

The solution of the deterministic equation

$$F_t(\xi) = 1 + \xi^{(k-1)}(t) + \int_0^t \xi^{(k)}(s) F_s(\xi)\, ds. \tag{1.4.7}$$

is given by

$$F_t(\xi) = [1 + (-1)^{k-1}\langle \delta_0^{(k-1)}, \xi\rangle] \cdot e^{(-1)^{k-1}\langle \delta_t^{(k-1)} - \delta_0^{(k-1)}, \xi\rangle}. \tag{1.4.8}$$

The right side of (1.4.8) is the S-transform of the right side of (1.4.3). Thus by injectivity of S-transform, we conclude that the solution of (1.4.2) is given by (1.4.3). This completes the proof of the claim.

EXAMPLE 1.4.2 Consider the following equation

$$X_t = \theta_t + \int_0^t : \dot{B}(s)^2 : \diamond X_s ds \tag{1.4.9}$$

where $: \dot{B}(s)^2 :=: \langle \cdot, \delta_s \rangle^2 := \langle : \cdot^{\otimes 2} :, \delta_s^{\otimes 2} \rangle$. Denote $f(s, \Phi) =: \dot{B}(s)^2 : \diamond\Phi$, then

$$S f(s, \Phi)(\xi) = \xi^2(s) S\Phi(\xi).$$

It is easy to check that $f(s, \Phi)$ satisfies conditions $(A1 - A3)$ for $\beta = 0$, then by Theorem 1.3.2, equation (1.4.9) has a unique solution X in $(\mathcal{S})^*$ for any $\theta \in \mathcal{A}_f$.

Let us write the solution explicitely. Let $F(t)(\xi) = SX_t(\xi)$, and $g(t)(\xi) = S\theta_t(\xi)$. Then we have

$$F(t)(\xi) = g(t)(\xi) + \int_0^t \xi^2(s) F(s)(\xi) ds.$$

As a function of t, $F(t)(\xi)$ is defined only almost everywhere for any given ξ, therefore it would be meaningless to differentiate it. But if we assume further that the function $t \mapsto g(t)(\xi)$ is differentiable for each given $\xi \in \mathcal{S}_c(\mathbf{R})$, and if we define

$$G_\xi(t) = g(t)(\xi) + \int_0^t \xi^2(s) F(s)(\xi) ds, \quad t \in [0, T], \tag{1.4.10}$$

then for almost all $t \in [0,T], G_\xi(t) = F(t)(\xi)$ for all $\xi \in \mathcal{S}_c(\mathbf{R})$. Moreover, for each ξ, the function $G_\xi(t)$ is differentiable and

$$G'_\xi(t) = g'(t)(\xi) + \xi^2(t)G_\xi(t), a.e.t \in [0,T]. \tag{1.4.11}$$

We first consider the following differential equation

$$G'_\xi(t) = \xi^2(t)G_\xi(t).$$

Its solution is given by

$$G_\xi(t) = Ce^{\int_0^t \xi^2(s)ds},$$

where C is a constant. Now write

$$G_\xi(t) = C(t)e^{\int_0^t \xi^2(s)ds}, \tag{1.4.12}$$

and differentiate both sides to get

$$G'_\xi(t) = C'(t)e^{\int_0^t \xi^2(s)ds} + C(t)e^{\int_0^t \xi^2(s)ds}\xi^2(t). \tag{1.4.13}$$

Combining (1.4.11), (1.4.12) and (1.4.13) we have

$$g'(t)(\xi) = C'(t)e^{\int_0^t \xi^2(s)ds}.$$

Solving this and noticing that $G_\xi(0) = C(0) = g(0)(\xi)$, we get

$$C(t) = g(t)(\xi)e^{-\int_0^t \xi^2(s)ds} + \int_0^t g(s)(\xi)\xi^2(s)e^{-\int_0^s \xi^2(u)du}ds.$$

Therefore

$$G_\xi(t) = g(t)(\xi) + \int_0^t g(s)(\xi)\xi^2(s)e^{\int_s^t \xi^2(u)du}ds, a.e.t \in [0,T].$$

Hence for almost all $t \in [0,T]$,

$$F(t)(\xi) = g(t)(\xi) + \int_0^t g(s)(\xi)\xi^2(s)e^{\int_s^t \xi^2(u)du}ds.$$

Since

$$|e^{\int_s^t \xi^2(u)du}| = |e^{\int_s^t \langle \xi, \delta_u \rangle^2 du}| \le |e^{\int_s^t |\xi|_p^2 |\delta_u|_{-p}^2 du} \le e^{T|\xi|_p^2},$$

for large p, the function $e^{\int_s^t \xi^2(u)du}, \xi \in \mathcal{S}_c(\mathbf{R})$, is a U-functional. By the characterization theorem in $(\mathcal{S})^*$, there exists a unique element $\Psi_{s,t} \in (\mathcal{S})^*$, such that $S\Psi_{s,t}(\xi) = e^{\int_s^t \xi^2(u)du}$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$. Thus the solution of equation (1.4.9) is given by

$$X_t = \theta_t + \int_0^t \theta_s \diamond : \dot{B}(s)^2 : \diamond \Psi_{s,t} ds, \quad t \in [0,T].$$

The following example has a solution in $(\mathcal{S})^*_\beta$ for $\beta = 1 - \frac{2}{n}$.

EXAMPLE 1.4.3 For $n \ge 3$, consider the following equation:

$$X_t = \theta_t + \int_0^t : \dot{B}(s)^n : \diamond X_s ds, \tag{1.4.14}$$

where $: \dot{B}(s)^n :=: \langle \cdot, \delta_s \rangle^n := \langle : \cdot^{\otimes n} :, \delta_s^{\otimes n} \rangle$.

Denote $f(s, \Phi) =: \dot{B}(s)^n : \diamond \Phi$. Then, $Sf(s,\Phi)(\xi) = \xi^n(s) S\Phi(\xi)$. It is easily seen that $f(s,\Phi)$ satisfies conditions $(A1 - A3)$ for $\beta = 1 - \frac{2}{n}$. Then by Theorem 1.3.2, equation (1.4.14) has a unique solution X in $(\mathcal{S})^*_{1-\frac{2}{n}}$ for any $\theta \in \mathcal{A}_f$.

Similar to Example 1.4.2, we have the following explicit expression for the solution:

$$X_t = \theta_t + \int_0^t \theta_s \diamond : \dot{B}(s)^n : \diamond \Psi_{s,t}^{(n)} ds \in (\mathcal{S})^*_{1-\frac{2}{n}},$$

where $\Psi_{s,t}^{(n)} \in (\mathcal{S})^*_{1-\frac{2}{n}}$ whose S-transform is given by $S\Psi_{s,t}^{(n)}(\xi) = e^{\int_s^t \xi^n(u)du}$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

The analogue of the finite dimensional Dirac's delta function in the white noise distribution theory is the Kubo-Yokoi delta function $\tilde{\delta}_x$, defined by

$$\langle\langle \tilde{\delta}_x, \varphi \rangle\rangle = \int_{\mathcal{S}_c'(\mathbf{R})} \varphi(y) d\delta_x(y) = \varphi(x), \quad \varphi \in (\mathcal{S}),$$

where δ_x is the delta measure on $\mathcal{S}_c'(\mathbf{R})$ at x. Since continuous versions of test functions are assumed (see Chapter 6 in [K2]), so that $\varphi(x)$ is meaningful. By equation (6.16) in [K2], the linear functional $\tilde{\delta}_x$ is continuous on $(\mathcal{S})$ and so $\tilde{\delta}_x \in (\mathcal{S})^*$ for each $x \in \mathcal{S}'_c(\mathbf{R})$.

EXAMPLE 1.4.4 Consider the following equation

$$X_t = 1 + \int_0^t \tilde{\delta}_{\delta_s} \diamond X_s ds, \quad t \in [0, T],$$

where $\tilde{\delta}_{\delta_s}$ is the Kubo-Yokoi delta function $\tilde{\delta}_x$ at $x = \delta_s$ for $s \in [0, T]$.

By Theorem 7.8 in [K2], the Kubo-Yokoi delta function $\tilde{\delta}_x$ at $x \in \mathcal{S}'_c(\mathbf{R})$ has the following Wiener-Itô decomposition

$$\tilde{\delta}_x = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, : x^{\otimes n} : \rangle. \tag{1.4.15}$$

Its S-transform is given by

$$S(\tilde{\delta}_x)(\xi) = e^{\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle}, \quad \xi \in \mathcal{S}_c(\mathbf{R}). \tag{1.4.16}$$

Let $f(s, \Phi) = \tilde{\delta}_{\delta_s} \diamond \Phi$. Then by (1.4.16) we get

$$Sf(s, \Phi)(\xi) = e^{\xi(s) - \frac{1}{2}\langle \xi, \xi \rangle} S\Phi(\xi).$$

It is easily seen that f does not satisfy the Lipschitz condition in Theorem 1.3.2. In fact, we will see that the equation has a unique solution in $\mathcal{M}^*$ instead of $(\mathcal{S})^*_\beta$ (see Example 2.3.3 below for details).

Chapter 2. White Noise Integration in $\mathcal{M}^*$

2.1. The Meyer-Yan Spaces

Example 1.4.4 is one of many equations that can not be solved in the spaces $(\mathcal{S})^*_\beta, 0 \leq \beta < 1$. Therefore we need to look for solutions in a larger space. The spaces $\mathcal{M}$ and $\mathcal{M}^*$ were first introduced by Meyer and Yan in [MY1, MY2]. Recently, Kondratiev and Streit [KS] have reconstructed the Meyer-Yan spaces of test functionals and distributions in a different way, and have developed a systematic theory for these spaces.

Recall that any $\varphi \in (L^2)$ has Wiener-Itô decomposition given by:

$$\varphi = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad f_n \in \hat{L}^2_c(\mathbb{R}^n).$$

For $p, q \geq 0$, define the following norms on (L^2):

$$\|\varphi\|_{\mathcal{M}_{p,q}} = \sqrt{\sum_{n=0}^{\infty} (n!)^2 2^{-nq} |f_n|_p^2}, \tag{2.1.1}$$

$$\|\Phi\|_{\mathcal{M}_{-p,-q}} = \sqrt{\sum_{n=0}^{\infty} 2^{nq} |F_n|_{-p}^2}. \tag{2.1.2}$$

Let

$$\mathcal{M}_{p,q} = \{\varphi \in (L^2) : \|\varphi\|_{\mathcal{M}_{p,q}} < \infty\}, \tag{2.1.3}$$

$$\mathcal{M}_{-p,-q} = \{\text{The completion of } (L^2) \text{ with respect to } \|\cdot\|_{\mathcal{M}_{-p,-q}}\}. \tag{2.1.4}$$

Then $\mathcal{M}_{-p,-q} = \mathcal{M}^*_{p,q}$, is the dual space of $\mathcal{M}_{p,q}$. Moreover for any fixed $p \geq 0, \mathcal{M}_{p,q_1} \subset \mathcal{M}_{p,q_2}$, for $q_1 \leq q_2$. Define $\mathcal{M}_p$ to be the inductive limit of the family of Hilbert spaces $\{\mathcal{M}_{p,q} : q \geq 0\}$, i.e.

$$\mathcal{M}_p = \bigcup_{q \geq 0} \mathcal{M}_{p,q}, \quad p \geq 0. \tag{2.1.5}$$

Then for any $p_2 \leq p_1, \mathcal{M}_{p_1} \subset \mathcal{M}_{p_2}$. Let $\mathcal{M}$ be the projective limit of $\{\mathcal{M}_p : p \geq 0\}$, i.e.

$$\mathcal{M} = \bigcap_{p \geq 0} \mathcal{M}_p = \bigcap_{p \geq 0} \bigcup_{q \geq 0} \mathcal{M}_{p,q}. \tag{2.1.6}$$

Let $\mathcal{M}_p^*$ be the dual space of $\mathcal{M}_p$ and $\mathcal{M}^*$ the dual space of $\mathcal{M}$. Then we have

$$\mathcal{M}^* = \bigcup_{p\geq 0}\mathcal{M}_p^* = \bigcup_{p\geq 0}\bigcap_{q\geq 0}\mathcal{M}_{p,q}^*. \tag{2.1.7}$$

This is the Kondratiev-Streit construction for the Meyer-Yan spaces $\mathcal{M}$ and $\mathcal{M}^*$. The norms here are different from the original norms given by Meyer and Yan, but they give the same topologies on $\mathcal{M}$ and $\mathcal{M}^*$. We have the following embedding relationship:

$$\mathcal{M} \subset (\mathcal{S})_\beta \subset (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* \subset (\mathcal{S})_\beta^* \subset \mathcal{M}^*.$$

Next we introduce the S-transform in $\mathcal{M}^*$. Recall that

$$: e^{\langle \cdot,\xi\rangle} := \sum_{n=0}^{\infty}\langle : \cdot^{\otimes n} :, \frac{\xi^{\otimes n}}{n!}\rangle, \quad \xi \in \mathcal{S}_c(\mathbf{R}). \tag{2.1.8}$$

For any $\xi \in \mathcal{S}_c(\mathbf{R})$ and $p \geq 0$, choose $q > 2\log_2(|\xi|_p)$. Then we have

$$\| : e^{\langle \cdot,\xi\rangle} : \|_{\mathcal{M}_{p,q}}^2 = \sum_{n=0}^{\infty}(n!)^2 2^{-nq}|\frac{\xi^{\otimes n}}{n!}|_p^2 = \sum_{n=0}^{\infty}(\frac{|\xi|_p^2}{2^q})^n < \infty. \tag{2.1.9}$$

This implies that $: e^{\langle \cdot,\xi\rangle} :\in \mathcal{M}$, for any $\xi \in \mathcal{S}_c(\mathbf{R})$. Thus for any $\Phi \in \mathcal{M}^*$, its S-transform can be defined by setting

$$S\Phi(\xi) = \ll \Phi, : e^{\langle \cdot,\xi\rangle} : \gg, \quad \xi \in \mathcal{S}_c(\mathbf{R}), \tag{2.1.10}$$

where $\ll \cdot,\cdot \gg$ is the natural pairing between $\mathcal{M}^*$ and $\mathcal{M}$. Similar to the Wiener-Itô decomposition for the distributions in $(\mathcal{S})_\beta^*$, we have:

Proposition 2.1.1 *Let $\Phi \in \mathcal{M}^*$. Then Φ has the following Wiener-Itô decomposition:*

$$\Phi(\cdot) = \sum_{n=0}^{\infty}\langle : \cdot^{\otimes n} :, F_n\rangle, \tag{2.1.11}$$

where $F_n \in \hat{\mathcal{S}}_c'(\mathbf{R}^n)$ satisfies

$$\|\Phi\|_{\mathcal{M}_{-p,-q}} = \sqrt{\sum_{n=0}^{\infty}2^{nq}|F_n|_{-p}^2} < \infty, \tag{2.1.12}$$

for some $p \geq 0$ and any $q \geq 0$.

PROOF. Let $\Phi \in \mathcal{M}^*$, then by definition, there exists $p \geq 0$ such that $\Phi \in \mathcal{M}^*_{p,q}$ for any $q \geq 0$. Let $\varphi_n(x) = \langle : x^{\otimes n} :, f_n \rangle$, where $f_n \in \hat{\mathcal{S}}_c(\mathbf{R}^n)$. Then $\varphi_n \in \mathcal{M}_{p,q}$ for any $p, q \geq 0$, and

$$\begin{aligned} | \ll \Phi, \varphi_n \gg | &\leq \|\Phi\|_{M_{-p,-q}} \|\varphi_n\|_{M_{p,q}} \\ &= \|\Phi\|_{M_{-p,-q}} n! 2^{-\frac{nq}{2}} |f_n|_p. \end{aligned}$$

Let $(L^2) = \bigoplus \mathcal{H}_n$ be the Wiener-Itô decomposition of (L^2), then the restriction Φ_n of Φ onto $\mathcal{M} \bigcap \mathcal{H}_n$ is in one-to-one correspondence with an element $F_n \in \hat{\mathcal{S}}'_{p,c}(\mathbf{R}^n)$, such that

$$\ll \Phi_n, \varphi_n \gg = n! \langle F_n, f_n \rangle. \tag{2.1.13}$$

Thus $\Phi_n(\cdot) = \langle : \cdot^{\otimes n} :, F_n \rangle$. Moreover,

$$\begin{aligned} \|\Phi_n\|_{M_{-p,-q}} &= \sup_{\varphi_n = \langle : x^{\otimes n} :, f_n \rangle, \|\varphi_n\|_{M_{p,q}} = 1} | \ll \Phi_n, \varphi_n \gg | \\ &= \sup_{\{f_n \in \hat{\mathcal{S}}_c(\mathbf{R}^n) : n! 2^{-\frac{nq}{2}} |f_n|_p = 1\}} n! |\langle F_n, f_n \rangle| \\ &= \sup_{\{f_n \in \hat{\mathcal{S}}_c(\mathbf{R}^n) : n! 2^{-\frac{nq}{2}} |f_n|_p = 1\}} (2^{\frac{nq}{2}} |F_n|_{-p})(n! 2^{-\frac{nq}{2}} |f_n|_p) \\ &= 2^{\frac{nq}{2}} |F_n|_{-p}. \end{aligned}$$

Since a general $\varphi \in \mathcal{M} \subset (\mathcal{S})$ can be expressed as

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad f_n \in \hat{\mathcal{S}}_c(\mathbf{R}^n),$$

we have

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} \ll \Phi_n, \varphi_n \gg = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \tag{2.1.14}$$

and

$$\|\Phi\|_{M_{-p,-q}} = \sqrt{\sum_{n=0}^{\infty} \|\Phi_n\|^2_{M_{-p,-q}}} = \sqrt{\sum_{n=0}^{\infty} 2^{nq} |F_n|^2_{-p}} < \infty. \tag{2.1.15}$$

Thus

$$\Phi(\cdot) = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, F_n \rangle,$$

which is the Wiener-Itô decomposition for $\Phi \in \mathcal{M}^*$. □

Let $\Phi \in \mathcal{M}^*$ have Wiener-Itô decomposition (2.1.11). Then from (2.1.8), (2.1.14) and definition (2.1.10),

$$S\Phi(\xi) = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathcal{S}_c(\mathbf{R}). \tag{2.1.16}$$

PROPOSITION 2.1.2 *Let $\Phi \in \mathcal{M}^*$. Then for any ξ and η in $\mathcal{S}_c(\mathbf{R})$, the function $S\Phi(z\xi + \eta), z \in \mathbf{C}$ is an entire function.*

PROOF. Let

$$\Phi \in \mathcal{M}^* = \bigcup_{p\geq 0} \bigcap_{q \geq 0} \mathcal{M}^*_{p,q}.$$

Then there exists $p \geq 0$ such that for all $q \geq 0$

$$\|\Phi\|^2_{\mathcal{M}_{-p,-q}} = \sum_{n=0}^{\infty} 2^{nq} |F_n|^2_{-p} < \infty. \tag{2.1.17}$$

Assume that Φ has Wiener-Itô decomposition (2.1.11), then by (2.1.16), we have for any $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, and any $z \in \mathbf{C}$,

$$S\Phi(z\xi + \eta) = \sum_{n=0}^{\infty} \langle F_n, (z\xi + \eta)^{\otimes n} \rangle. \tag{2.1.18}$$

For any $r \geq 0$, any $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, and any $z \in \mathbf{C}$ with $|z| \leq r$, choose $q > \log_2(|\eta|_p^2 + r^2|\xi|_p^2)$. Then $\sum_{n=0}^{\infty}(r|\xi|_p)^{2n}2^{-nq} < \infty$, and $\sum_{n=0}^{\infty} |\eta|_p^{2n} 2^{-nq} < \infty$. From this and (2.1.17) we have

$$\begin{aligned} \sum_{n=0}^{\infty} |F_n|_{-p} |\eta|_p^n &= \sum_{n=0}^{\infty} |F_n|_{-p} 2^{\frac{nq}{2}} \cdot 2^{-\frac{nq}{2}} |\eta|_p^n \\ &\leq \sqrt{\sum_{n=0}^{\infty} |F_n|^2_{-p} 2^{nq}} \cdot \sqrt{\sum_{n=0}^{\infty} |\eta|_p^{2n} 2^{-nq}} < \infty, \end{aligned}$$

and

$$\sum_{n=0}^{\infty} |F_n|_{-p}|z|^n|\xi|_p^n = \sum_{n=0}^{\infty} |F_n|_{-p}(r|\xi|_p)^n$$

$$\leq \sqrt{\sum_{n=0}^{\infty} |F_n|_{-p}^2 2^{nq}} \cdot \sqrt{\sum_{n=0}^{\infty} (r|\xi|_p)^{2n} 2^{-nq}} < \infty.$$

Furthermore, we have

$$|\sum_{n=0}^{\infty} \langle F_n, (z\xi + \eta)^{\otimes n}\rangle| \leq \sum_{n=0}^{\infty} |\langle F_n, (z\xi + \eta)^{\otimes n}\rangle|$$

$$\leq \sum_{n=0}^{\infty} |F_n|_{-p}|z\xi + \eta|_p^n = \sum_{n=0}^{\infty} |F_n|_{-p}(|z|^n|\xi|_p^n + |\eta|_p^n).$$

Therefore the series in (2.1.18) converges uniformly on any compact subset of $\mathbf{C}$. The proposition follows from this and the fact that each function $\langle F_n, (z\xi + \eta)^{\otimes n}\rangle$, $z \in \mathbf{C}$ is entire. □

The following characterization theorem can be found in [KS] stated in a different way.

THEOREM 2.1.3 *If $\Phi \in \mathcal{M}^*$, then the function $F(\xi) = S\Phi(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$, satisfies the following two conditions:*

(1) For any $\xi, \eta \in \mathcal{S}_c(\mathbf{R}), z \mapsto F(z\xi + \eta)$, is an entire function on $\mathbf{C}$.

(2) For any $R \geq 0$, there exist $p \geq 0$, and a nonnegative constant C depending on p, R such that, $M(R,\xi) \equiv \sup_{z\in\mathbf{C}, |z|=R} |F(z\xi)| \leq C$, for any $\xi \in \mathcal{S}_c(\mathbf{R})$, with $|\xi|_p \leq 1$.

Conversely, if a function $F(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$, satisfies the above conditions, then there exists a unique $\Phi \in \mathcal{M}^*$, such that $F(\xi) = S\Phi(\xi)$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

PROOF. Let $\Phi \in \mathcal{M}^*$, and $F(\xi) = S\Phi(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$. Then Proposotion 2.1.2 implies (1), and the estimates (2.1.19) through (2.1.21) imply (2). The proof for the other direction can be found in [KS]. □

DEFINITION 2.1.4 *A function satisfying the above two conditions is called a quasi-U-functional.*

Theorem 2.1.3 implies that for any quasi-U-functional $G(\xi)$ there exists a unique $\Phi \in \mathcal{M}^*$ such that $S\Phi(\xi) = G(\xi)$ for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

The following example gives us a quasi-U-functional which is not a U-functional (see Example 6.4, Poisson Noise [KS]).

Example 2.1.5. For $\xi \in \mathcal{S}_c(\mathbf{R})$, let

$$F(\xi) = \exp[\int_0^T e^{\xi(s)} ds].$$

Then for any $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, and any $z \in \mathbf{C}$,

$$\begin{aligned} F(z\xi + \eta) &= \exp[\int_0^T e^{z\xi(s)+\eta(s)} ds] \\ &= \exp[\int_0^T e^{z\langle \xi, \delta_s \rangle + \langle \eta, \delta_s \rangle} ds] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\int_0^T e^{z\langle \xi, \delta_s \rangle + \langle \eta, \delta_s \rangle} ds)^n. \end{aligned}$$

For $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, each function $\frac{1}{n!}(\int_0^T e^{z\langle \xi, \delta_s \rangle + \langle \eta, \delta_s \rangle} ds)^n, z \in \mathbf{C}$, is entire for $n \geq 0$. Moreover, we can choose $p > 0$ such that $|\delta_s|_{-p} \leq 1$ for all s. Thus for any $R \geq 0$, any $z \in \mathbf{C}$ with $|z| \leq R$, we have

$$\begin{aligned} & |\frac{1}{n!} (\int_0^T e^{z\langle \xi, \delta_s \rangle + \langle \eta, \delta_s \rangle} ds)^n| \\ \leq & \frac{1}{n!} (\int_0^T e^{|z||\xi|_p |\delta_s|_{-p} + |\eta|_p |\delta_s|_{-p}} ds)^n \\ \leq & \frac{1}{n!} (\int_0^T e^{|z||\xi|_p + |\eta|_p} ds)^n \\ \leq & \frac{T^n}{n!} e^{n(R|\xi|_p + |\eta|_p)} = \frac{a^n}{n!}, \end{aligned}$$

where $a = Te^{R|\xi|_p + |\eta|_p}$ is a constant for given ξ and η. This implies that the series $\sum_{n=0}^{\infty} \frac{1}{n!}(\int_0^T e^{z\langle \xi, \delta_s \rangle + \langle \eta, \delta_s \rangle} ds)^n$ converges uniformly on any compact subset of $\mathbf{C}$. Thus the function $F(z\xi + \eta), z \in \mathbf{C}$, is entire. On the other hand, for any $R \geq 0$, by the above argument, there exists $p \geq 0$ such that

$$\begin{aligned} & |F(z\xi)| = |\sum_{n=0}^{\infty} \frac{1}{n!} (\int_0^T e^{z\langle \xi, \delta_s \rangle} ds)^n| \\ \leq & \sum_{n=0}^{\infty} \frac{T^n}{n!} e^{nR|\xi|_p} = \exp[Te^{R|\xi|_p}], \end{aligned}$$

for all $z \in \mathbf{C}, |z| = R$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$. Thus

$$M(R,\xi) \equiv \sup_{z\in\mathbf{C},|z|=R} |F(z\xi)| \leq \exp[Te^{R}]$$

for all $\xi \in \mathcal{S}_c(\mathbf{R})$ with $|\xi|_p \leq 1$. Therefore $F(\xi)$ is a quasi-U-functional. But it is not a U-functional since it does not satisfy the growth condition of a U-functional.

2.2. White Noise Integrals

Recall that in [HP], a function f from $[0,T]$ to a separable Banach space is Bochner integrable if and only if it is weakly measurable and $\|f(\cdot)\| \in L^1[0,T]$. The space $(\mathcal{S})^*_\beta$ is not a Banach space, but $(\mathcal{S})^*_\beta = \bigcup_{p\geq 0}(\mathcal{S})^*_{p,\beta}$, and each $(\mathcal{S})^*_{p,\beta}$ is a separable Hilbert space. Thus in Chapter 1, a function $\Phi : [0,T] \to (\mathcal{S})^*_{p,\beta}$ is defined to be Bochner integrable if it satisfies the following conditions:

(1) Φ is weakly measurable.

(2) There exists a constant $p \geq 0$ such that $\Phi(s) \in (\mathcal{S})^*_{p,\beta}$ for almost all $s \in [0,T]$, and the function $\|\Phi(\cdot)\|_{-p,-\beta} \in L^1([0,T])$.

But the space $\mathcal{M}^*$ has more complicated structure, therefore we only extend the definition of Pettis integral to this space.

DEFINITION 2.2.1 *A function $\Phi : [0,T] \to \mathcal{M}^*$ is called Pettis integrable if*

(1) Φ is weakly measurable.

(2) For any $\varphi \in \mathcal{M}$, $\ll \Phi(\cdot), \varphi \gg \in L^1[0,T]$.

(3) For any $t \in [0,T]$, there exists a unique generalized function $\int_0^t \Phi_s ds \in \mathcal{M}^*$ such that

$$\ll \int_0^t \Phi_s ds, \varphi \gg = \int_0^t \ll \Phi_s, \varphi \gg ds, \quad \forall \varphi \in \mathcal{M}.$$

From the white noise view point, it is natural to define the integral $\int_0^t \Phi_s ds$ by using S-transform, i.e. we require the following conditions for Φ:

(a) The function $s \mapsto S\Phi_s(\xi)$ is measurable on [0, T] for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(b) The function $s \mapsto S\Phi_s(\xi)$ is integrable on [0, T] for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(c) For any $t \in [0,T]$, the function $\int_0^t S\Phi_s(\xi)ds, \xi \in \mathcal{S}_c(\mathbf{R})$ is the S-transform of a generalized function in $\mathcal{M}^*$, denoted by $\int_0^t \Phi_s ds$, i.e.

$$S(\int_0^t \Phi_s ds)(\xi) = \int_0^t S\Phi_s(\xi)ds, \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}).$$

Note that condition (a) implies condition (1). For each t, from (c) we see that there exists a generalized function in $\mathcal{M}^*$, denoted by $\int_0^t \Phi_s ds$, such that

$$\ll \int_0^t \Phi_s ds, \varphi_\xi \gg = \int_0^t \ll \Phi_s, \varphi_\xi \gg ds, \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}),$$

where $\varphi_\xi =: e^{\langle \cdot, \xi \rangle} :$. Define

$$T\varphi = \ll \int_0^t \Phi_s ds, \varphi \gg, \quad \varphi \in \mathcal{M}.$$

Then T is a linear continuous functional on $\mathcal{M}$ satisfying

$$T\varphi_\xi = \int_0^t \ll \Phi_s, \varphi_\xi \gg ds, \quad \forall \xi \in \mathcal{S}_c(\mathbf{R}).$$

For any $\varphi \in \mathcal{M}$, since the linear span V of the set $\{\varphi_\xi; \xi \in \mathcal{S}_c(\mathbf{R})\}$ is dense in $\mathcal{M}$, we can approximate φ by a sequence φ_n from V, therefore we get, for any $s \in [0,T]$,

$$\ll \Phi_s, \varphi_n \gg \to \ll \Phi_s, \varphi \gg, (n \to \infty),$$

and

$$\int_0^t \ll \Phi_s, \varphi_n \gg = \ll \int_0^t \Phi_s ds, \varphi_n \gg \to \ll \int_0^t \Phi_s ds, \varphi \gg, (n \to \infty).$$

This implies that the function $\ll \Phi_s, \varphi \gg, s \in [0,T]$, is Lebesgue integrable, and the following equality holds for all $\varphi \in \mathcal{M}$:

$$\int_0^t \ll \Phi_s, \varphi \gg ds = \ll \int_0^t \Phi_s ds, \varphi \gg, \quad \forall t \in [0,T].$$

Thus Φ is Pettis integrable by Definition 2.2.1, and so conditions (a)-(c) are equivalent to conditions (1)-(3) since the other direction is obvious. Therefore a generalized function is Pettis integrable if and only if it satisfies conditions (a)-(c).

The following simple lemma is useful in applications since the conditions are easier to verify than conditions (1)-(3) or conditions (a)-(c).

LEMMA 2.2.2 *A generalized function $\Phi : [0,T] \to \mathcal{M}^*$ is Pettis integrable if and only if it satisfies the conditions:*

(G1) (Measurability Condition) The function $s \mapsto S\Phi_s(\xi)$ is measurable on [0, T] for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(G2) (Growth Condition) There exist a nonnegative number K, and a nonnegative quasi-U-functional $G(\xi)$, such that for all $t \in [0,T]$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$,

$$\int_0^t |S\Phi_s(\xi)|ds \leq KG(\xi). \tag{2.2.1}$$

PROOF. It is obvious that conditions (1), (a) and (G1) are equivalent, and conditions (2) and (3) imply (G2).

On the other hand, (G2) implies that $S\Phi_s(\xi), s \in [0,T]$, is integrable on [0, T] for all $\xi \in \mathcal{S}_c(\mathbf{R})$. That is, condition (b) is satisfied. Denote $F_t(\xi) = \int_0^t S\Phi_s(\xi)ds$. From (2.2.1) we have, for all $z \in \mathbf{C}$, and all $\xi \in \mathcal{S}_c(\mathbf{R})$,

$$|F_t(z\xi)| \leq \int_0^t |S\Phi_s(z\xi)|ds \leq KG(z\xi).$$

Since $G(\xi)$ is a quasi-U-functional, then it is easily seen that for any $R \geq 0$ there exist $p \geq 0$, and a constant C depending on R and p, such that

$$M(R,\xi) \equiv \sup_{z\in\mathbf{C},|z|=R} F_t(z\xi) \leq KC,$$

for all $\xi \in \mathcal{S}_c(\mathbf{R})$ with $|\xi|_p \leq 1$. Furthermore, by using Morera's theorem we can check that, for any fixed $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, the function $F_t(z\xi + \eta), z \in \mathbf{C}$ is entire on C. Therefore $F_t(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$, is a quasi-U-functional, and so by Theorem 2.1.3 there exists a unique generalized function in $\mathcal{M}^*$, denoted by $\int_0^t \Phi_s ds$, such that

$$S(\int_0^t \Phi_s ds)(\xi) = \int_0^t S\Phi_s(\xi)ds, \quad \xi \in \mathcal{S}_c(\mathbf{R}).$$

Hence condition (c) is satisfied. Thus Φ is Pettis integrable, and Lemma 2.2.2 is proved. □

Let $f : [0,T] \times \mathcal{M}^* \to \mathcal{M}^*$ satisfy the following conditions:

(B1)(Measurability Condition) The function $Sf(s,\Phi_s)(\xi), s \in [0,T]$, is measurable for each $\xi \in \mathcal{S}_c(\mathbf{R})$, whenever $S\Phi_s(\xi), s \in [0,T]$, is.

(B2) (Growth Condition) For all $\Phi \in \mathcal{M}^*, \xi \in \mathcal{S}_c(\mathbf{R})$, and a.e. $s \in [0,T]$,

$$|Sf(s,\Phi)(\xi)| \leq H(s)G(\xi)(1 + |S\Phi(\xi)|),$$

where $H(\cdot) \in L^1[0,T]$, and G is a quasi-U-functional.

(B3) (Lipschitz Condition) For all $\Phi, \Psi \in \mathcal{M}^*, \xi \in \mathcal{S}_c(\mathbf{R})$, and a.e. $s \in [0,T]$,

$$|Sf(s,\Phi)(\xi) - Sf(s,\Psi)(\xi)| \leq L(s,\xi)|S\Phi(\xi) - S\Psi(\xi)|,$$

where L is a nonnegative function satisfying $\int_0^T L(s,\xi)ds \leq G(\xi)$.

Let $\Phi : [0,T] \to \mathcal{M}^*$ satisfy the following conditions:

(D1)(Measurability Condition) The function $s \mapsto S\Phi_s(\xi)$ is measurable on [0, T] for each $\xi \in \mathcal{S}_c(\mathbf{R})$.

(D2) (Growth Condition) There exist a quasi-U-functional $G_1(\xi)$, and a function $K_0(\cdot) \in L^1[0,T]$, with $K_0(\cdot)H(\cdot) \in L^1[0,T]$, and $K_0(\cdot)L(\cdot,\xi) \in L^1[0,T]$, such that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and a.e. $s \in [0,T]$,

$$|S\Phi_s(\xi)| \leq K_0(s)G_1(\xi).$$

Denote $\mathcal{B}_f = \{\Phi : [0,T] \to \mathcal{M}^*, \text{satisfying } (D1) \text{ and } (D2)\}$.

LEMMA 2.2.3 *If f satisfies conditions (B1-B3) and $\Phi \in \mathcal{B}_f$, then $f(s,\Phi_s), s \in [0,T]$, is Pettis integrable and the function $\int_0^t f(s,\Phi_s)ds, t \in [0,T]$, is also in $\mathcal{B}_f$.*

PROOF. Conditions (B1) and (D1) imply that the function $s \mapsto Sf(s,\Phi_s)(\xi)$ is measurable on [0, T]. Thus condition (G1) is satisfied. Moreover

$$\begin{aligned}|Sf(s,\Phi_s)(\xi)| &\leq H(s)G(\xi)(1+|S\Phi_s(\xi)|)\\ &\leq H(s)G(\xi)(1+K_0(s)G_1(\xi))\\ &\leq H_1(s)G_2(\xi),\end{aligned}$$

where $H_1(s) = H(s) + H(s)K_0(s) \in L^1[0,T]$, and $G_2(\xi) = G(\xi) + G(\xi)G_1(\xi)$ is a quasi-U-functional. Therefore we have, for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and all $t \in [0,T]$

$$\int_0^t |Sf(s,\Phi_s)(\xi)|ds \leq \int_0^t H_1(s)dsG_2(\xi) \leq \int_0^T H_1(s)dsG_2(\xi).$$

Thus (G2) is satisfied, and so by Lemma 2.2.2, the function $f(s,\Phi_s), s \in [0,T]$, is Pettis integrable. Furthermore,

$$t \mapsto S(\int_0^t f(s,\Phi_s)ds)(\xi) = \int_0^t Sf(s,\Phi_s)(\xi)ds$$

is absolutely continuous, thus measurable on [0, T]. Hence $\int_0^t f(s,\Phi_s)ds, t \in [0,T]$, is in $\mathcal{B}_f$. □

2.3. White Noise Integral Equations

Consider the following white noise integral equation in $\mathcal{M}^*$:

$$X_t = \theta_t + \int_0^t f(s,X_s)ds, \tag{2.3.1}$$

where $\theta : [0,T] \to \mathcal{M}^*$ and $f : [0,T] \times \mathcal{M}^* \to \mathcal{M}^*$ are generalized functions, and $\int_0^t f(s,X_s)ds$ is a white noise integral in the Pettis sense.

DEFINITION 2.3.1 *A generalized function $X : [0,T] \to \mathcal{M}^*$ is called a solution of equation (2.3.1) on $[0,T]$ if it satisfies the following conditions:*

(1) (Measurability Condition) X is weakly measurable.

(2) (Integrability Condition) The function $s \mapsto f(s,X_s), s \in [0,T]$, is Pettis integrable.

(3) For any $\varphi \in \mathcal{M}$, and a.e.$t \in [0,T]$, the following equality holds:

$$\langle\langle X_t,\varphi\rangle\rangle = \langle\langle \theta_t,\varphi\rangle\rangle + \int_0^t \langle\langle f(s,X_s),\varphi\rangle\rangle\, ds.$$

THEOREM 2.3.2 *Let $f : [0,T] \times \mathcal{M}^* \to \mathcal{M}^*$ be a generalized function that maps $[0,T] \times (\mathcal{S})^*_\beta$ into $(\mathcal{S})^*_\beta$, and satisfies the following conditions:*

(F1) (Measurability Condition) The function $s \mapsto Sf(s,\Phi)(\xi)$ is measurable on [0, T] for any $\Phi \in \mathcal{M}^*$ and any $\xi \in \mathcal{S}_c(\mathbf{R})$.

(F2) (Growth Condition) There exist a nonnegative function $H(\cdot) \in L^1[0,T]$, and nonnegative constants a, p, such that for all $\Phi \in \mathcal{M}^*$, $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|Sf(s,\Phi)(\xi)| \le H(s)\exp[a|\xi|_p^{\frac{2}{1-\beta}}](1 + |S\Phi(\xi)|).$$

(F3) (Lipschitz Condition) There exists a function $L : [0,T] \times \mathcal{S}_c(\mathbf{R}) \to \mathbf{R}^+$ satisfying $\int_0^T L(s,\xi)ds \le b(1 + \exp[|\xi|_r^{\frac{2}{1-\beta}}])$ for some nonnegative constants b and r, such that, for all $\Phi, \Psi \in \mathcal{M}^*$, $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|Sf(s,\Phi)(\xi) - Sf(s,\Psi)(\xi)| \le L(s,\xi)|S\Phi(\xi) - S\Psi(\xi)|.$$

Let $\theta \in \mathcal{A}_f$ (see Chapter 1, in particular, $\theta_t \in (\mathcal{S})^*_\beta$). Then equation (2.3.1) has a unique solution $X : [0,T] \to \mathcal{M}^*$ satisfying condition (D2) (see Section 2.2).

PROOF. Uniqueness: Similar to the proof of Theorem 1.3.2.

To show existence we use the iteration method. For $t \in [0,T]$, we define $X_t^{(0)} = \theta_t$, and $X_t^{(n+1)} = \theta_t + \int_0^t f(s, X_s^{(n)})\,ds$, for $n \ge 0$. It is obvious that $(F1)$ and $(F2)$ imply (A1) and (A2) respectively. Moreover, when we replace (A3) by $(F3)$ in Lemma 1.2.3 and 1.2.4, all the conclusions there still hold. Hence by Lemma 1.2.4, for $n \ge 0$, the white noise integral $\int_0^t f(s, X_s^{(n)})ds$ exists in the Pettis sense, and $\{X^{(n)}\}_{n\ge 0} \subset \mathcal{A}_f$. Let $F_t^{(n)}(\xi) = SX_t^{(n)}(\xi)$. By conditions (A2), (A3) for f, and (C2) for θ, there exist positive constants b, r, c_1, a_1 and p_1 such that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)| \le \frac{1}{n!}[b(1 + \exp[|\xi|_r^{\frac{2}{1-\beta}}])]^n c_1 \exp[a_1|\xi|_{p_1}^{\frac{2}{1-\beta}}].$$

Thus there exists a functional $F_t(\xi)$, such that, for all $\xi \in \mathcal{S}_c(\mathbf{R})$, $F_t^{(n)}(\xi) \to F_t(\xi)$ uniformly for almost all t. Moreover, by the above inequality and (C2) for θ, there exists $K_0(\cdot) \in L^1[0,T]$ satisfying $K_0(\cdot)H(\cdot) \in L^1[0,T], K_0(\cdot)L(\cdot,\xi) \in L^1[0,T]$, and

$$|F_t(\xi)| \leq H_1(t) \exp[a_1|\xi|_{p_1}^{\frac{2}{1-\beta}} + b(1 + \exp[|\xi|_r^{\frac{2}{1-\beta}}])], \tag{2.3.2}$$

where $H_1 = K_0$+constant. Therefore $F_t(\xi)$ is a quasi-U-functional for almost all $t \in [0,T]$. By Theorem 2.1.3, there exists $X_t \in \mathcal{M}^*$ such that $SX_t(\xi) = F_t(\xi)$ for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$.

We now need to show that X_t satisfies the three conditions of Definition 2.3.1. Pick and fix any $\xi \in \mathcal{S}_c(\mathbf{R})$. Since $F_t^{(n)}(\xi)$ converges to $F_t(\xi)$, and for each n, $F_t^{(n)}(\xi), t \in [0,T]$, is measurable on [0, T], $F_t(\xi), t \in [0,T]$, is measurable on [0, T]. Since the set $\{: e^{\langle\cdot,\xi\rangle} :; \xi \in \mathcal{S}_c(\mathbf{R})\}$ is dense in $\mathcal{M}$, X_t is weakly measurable by a limiting argument. Thus X satisfies the measurability condition in Definition 2.3.1. The integrability of $H_1 H$ on [0, T] and (2.3.2) imply that X satisfies the growth condition (C2). Thus $X \in \mathcal{B}_f$. The integrability condition in Definition 2.3.1 follows from this and Lemma 2.2.3. The proof of the third condition is similar to that in the proof of Theorem 1.3.2. Thus X is a solution of equation (2.3.1), and so Theorem 2.3.2 is proved. □

THEOREM 2.3.3 *Let f satisfy conditions (B1)-(B3) and $\theta \in \mathcal{B}_f$. Then equation (2.3.1) has a unique solution $X : [0,T] \to \mathcal{M}^*$ satisfying condition (D2).*

PROOF. Uniqueness: Similar to the proof in Theorem 1.3.2.
Existence: we use the iteration method again. For $t \in [0,T]$, let

$$\begin{aligned} X_t^{(0)} &= \theta_t, \\ X_t^{(n+1)} &= \theta_t + \int_0^t f(s, X_s^{(n)})\, ds, \quad n \geq 0. \end{aligned}$$

By Lemma 2.2.3 all the white noise integrals $\int_0^t f(s, X_s^{(n)})\, ds, n \geq 0$, exist in the Pettis sense and $\{X^{(n)}\}_{n\geq 0} \subset \mathcal{B}_f$.

Let $F_t^{(n)}(\xi) = SX_t^{(n)}(\xi)$, for all $\xi \in \mathcal{S}_c(\mathbf{R})$. Combining the growth conditions $(B2)$ and $(D2)$, we have for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and almost all $t \in [0,T]$,

$$|Sf(s,\theta_s)(\xi)| \leq K_1(s)G_2(\xi),$$

where $K_1(s) = H(s) + H(s)K_0(s) \in L^1[0,T]$, and $G_2(\xi) = G(\xi) + G(\xi)G_1(\xi)$ is a quasi-U-functional. Thus by conditions (B3) and (D2) , there exists a null set N_0 such that, for $\xi \in \mathcal{S}_c(\mathbf{R})$, and $t \notin N_0$,

$$\begin{aligned}
&|F_t^{(1)}(\xi) - F_t^{(0)}(\xi)| = |\int_0^t Sf(s,\theta_s)(\xi)\,ds| \\
&\leq G_2(\xi) \cdot \int_0^T K_1(s)\,ds = c_1 G_2(\xi),
\end{aligned}$$

where $c_1 = \int_0^T K_1(s)\,ds$. Note that c_1 and G_2 depend only on f and θ. Thus for each $n \geq 0$, by Lipschitz condition $(B3)$ and the induction method, we have a null set A_n such that, for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and $t \notin A_n$,

$$\begin{aligned}
|F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)| &\leq \int_0^t L(s,\xi)|F_s^{(n)}(\xi) - F_s^{(n-1)}(\xi)|\,ds \\
&\leq \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} L(s_1,\xi)L(s_2,\xi)\cdots L(s_n,\xi) \\
&\quad ds_1 ds_2 \cdots ds_n \cdot \sup_{s \notin N_0} |F_s^{(1)}(\xi) - F_s^{(0)}(\xi)| \\
&\leq \frac{1}{n!}\left(\int_0^t L(s,\xi)\,ds\right)^n c_1 G_2(\xi) \\
&\leq \frac{[G(\xi)]^n}{n!} c_1 G_2(\xi).
\end{aligned}$$

Hence, for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and $t \notin N_1 \equiv \cup_{n\geq 0} A_n$,

$$\sum_{n=0}^{\infty} |F_t^{(n+1)}(\xi) - F_t^{(n)}(\xi)| \leq c_1 G_2(\xi) e^{G(\xi)}.$$

This means that for all $\xi \in \mathcal{S}_c(\mathbf{R})$ and almost all $t \in [0,T]$,

$$\begin{aligned}
|F_t^{(n)}(\xi)| &= |F_t^{(0)}(\xi) + \sum_{m=0}^{n-1} [F_t^{(m+1)}(\xi) - F_t^{(m)}(\xi)]| \\
&\leq |F_t^{(0)}(\xi)| + \sum_{m=0}^{\infty} |F_t^{(m+1)}(\xi) - F_t^{(m)}(\xi)|
\end{aligned}$$

$$\begin{aligned}
&\le |S\theta_t(\xi)| + c_1 G_2(\xi) e^{G(\xi)} \\
&\le K_0(t)G(\xi) + c_1 G_2(\xi) e^{G(\xi)} \\
&\le H_1(t) G_3(\xi),
\end{aligned}$$

where $G_3(\xi) = G(\xi) + G_2(\xi)e^{G(\xi)}, \xi \in \mathcal{S}_c(\mathbf{R})$, is a quasi-U-functional, and $H_1(\cdot) = K_0(\cdot) + c_1 \in L^1[0,T]$. The functions $H_1(t)$ and $G_3(\xi)$ are independent of n. Thus for all $\xi \in \mathcal{S}_c(\mathbf{R})$, the sequence $F_t^{(n)}(\xi)$ converges to a limit $F_t(\xi)$ uniformly for a.e.t, and

$$|F_t(\xi)| \le H_1(t) G_3(\xi). \tag{2.3.3}$$

Moreover, for any fixed ξ and η in $\mathcal{S}_c(\mathbf{R})$, $F_t^{(n)}(z\xi+\eta)$ converges uniformly on any compact subset of $\mathbf{C}$ to $F_t(z\xi+\eta)$. Therefore the map $z \mapsto F_t(z\xi+\eta), z \in \mathbf{C}$, is entire. This and inequality (2.3.3) imply that $F_t(\xi), \xi \in \mathcal{S}_c(\mathbf{R})$, is a quasi-U-functional. Thus by Theorem 2.1.3 there exists $X_t \in \mathcal{M}^*$ such that $SX_t(\xi) = F_t(\xi)$ for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

Fix $\xi \in \mathcal{S}_c(\mathbf{R})$. Because $F_t^{(n)}(\xi)$ converges to $F_t(\xi)$, and each function $t \mapsto F_t^{(n)}(\xi)$ is measurable on [0, T], the function $t \mapsto F_t(\xi)$ is measurable on [0, T]. The weak measurability of X followes from this and the denseness of the set $\{: e^{\langle\cdot,\xi\rangle} :; \xi \in \mathcal{S}_c(\mathbf{R})\}$ in $\mathcal{M}$. From the measurability of $F_t(\xi), t \in [0,T]$, and (2.3.3), we conclude that $X \in \mathcal{B}_f$. This implies by Lemma 2.2.3 that $f(s, X_s), s \in [0,T]$, is Pettis integrable. Thus X satisfies the integrability condition in Definition 1.3.1.

Following a similar argument as that in the proof of Theorem 1.3.2, we can check condition (3) in Definition 2.3.1. That is, for almost all $t \in [0,T]$ and all $\varphi \in \mathcal{M}$,

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \int_0^t \ll f(s, X_s)ds, \varphi \gg ds.$$

Therefore X is a solution of equation (2.3.1), and the theorem is proved. □

EXAMPLE 2.3.3 Consider Example 1.4.4 again, i.e. the following equation:

$$X_t = \theta_t + \int_0^t \tilde{\delta}_{\delta_s} \diamond X_s ds.$$

It is easily seen that conditions $(F1)$-$(F3)$ in Theorem 2.3.2 are satisfied. Thus the equation has a unique solution X in $\mathcal{M}^*$. Using a computation similar to the one in Example 1.4.2, we can write the solution in the following explicit form:

$$X_t = \theta_t + \int_0^t \theta_u \diamond :e^{\dot{B}(u)}: \diamond \Psi_{t,u} du.$$

Here $\Psi_{t,u}$ is in $\mathcal{M}^*$ with S-transform $\exp[\int_u^t e^{\xi(s)} ds]$. The latter is indeed a quasi-U-functional but not a U-functional (see Example 2.1.5). Therefore $X \in \mathcal{M}^*$, and $X \notin (\mathcal{S})^*_\beta$, for any $\beta \in [0, 1)$.

Chapter 3. Some Applications

3.1. Equations Associated with Integral Kernel Operators

For $F \in L^2(\mathbf{R}^+)$, define:

$$A_s = \int_{-\infty}^{s} F(s-u)\partial_u du, \quad E_s = \exp(A_s).$$

Let A_s^* and E_s^* be their duals, respectively. These operators were first suggested by Oosawa et. al. [OTK], and then studied by Hida [H].

In this section we study the following white noise integral equations :

$$X_t = \theta_t + \int_0^t A_s^* X_s ds, \quad t \in [0,T]. \tag{3.1.1}$$

$$X_t = \theta_t + \int_0^t E_s^* X_s ds, \quad t \in [0,T]. \tag{3.1.2}$$

PROPOSITION 3.1.1 *For any $s \in [0,T]$, the function $A_s^*\Phi, \Phi \in (\mathcal{S})^*$ is continuous from $(\mathcal{S})^*$ into itself. Moreover, put*

$$\kappa_s(u) = F(s-u), \quad \textit{for} \quad u \le s; \quad 0, \quad \textit{otherwise}. \tag{3.1.3}$$

Then $\kappa_s \in L^2(\mathbf{R})$ and

$$A_s^*\Phi = \langle \cdot, \kappa_s \rangle \diamond \Phi. \tag{3.1.4}$$

PROOF. For any $\Phi \in (\mathcal{S})^*$, $s \in [0,T]$, and $\xi \in \mathcal{S}_c(\mathbf{R})$, we have

$$\begin{aligned} S(A_s^*\Phi)(\xi) &= S(\int_{-\infty}^{s} F(s-u)\partial_u^*\Phi)(\xi) \\ &= \int_{-\infty}^{s} F(s-u)\xi(u)du S\Phi(\xi) \\ &= \langle \kappa_s, \xi \rangle S\Phi(\xi) = S(\langle \cdot, \kappa_s \rangle \diamond \Phi)(\xi). \end{aligned}$$

This implies (3.1.4) since S-transform is injective. Furthermore, since

$$|\kappa_t|_0^2 = \int_{-\infty}^{\infty} |\kappa_t(u)|^2 du = \int_{-\infty}^{t} |F(t-u)|^2 du = \int_0^{\infty} F^2(u)du = |F|_0^2 < \infty,$$

we have $\kappa_s \in L^2(\mathbf{R})$. Thus $\langle \cdot, \kappa_s \rangle \in (L^2)$ for all s. Combining this and (3.1.4) we see that $A_s^*\Phi, \Phi \in (\mathcal{S})^*$ is continuous from $(\mathcal{S})^*$ into itself. □

In order to obtain a similar result for E_s^*, we need the following lemmas.

LEMMA 3.1.2 *For $\kappa \in \mathcal{S}'(\mathbf{R}^k)$, define*

$$\Xi_{0,k}(\kappa) = \int_{\mathbf{R}^k} \kappa(t_1, \cdots, t_k)\partial_{t_1} \cdots \partial_{t_k} dt_1 \cdots dt_k. \tag{3.1.5}$$

Then $\Xi_{0,k}(\kappa)$ is a continuous linear operator from $(\mathcal{S})$ into itself. Moreover, for any $p \geq 0, q > 0$, and any $\varphi \in (\mathcal{S})_{p+q}$,

$$\|\Xi_{0,k}(\kappa)\varphi\|_p \leq a_{\frac{q}{2}} |\kappa|_{-(p+q)} \|\varphi\|_{p+q},$$

where

$$a_{\frac{q}{2}} = 2^{\frac{q}{2}} k^{\frac{k}{2}} [\frac{2^q}{2qe(log2)}]^{\frac{k}{2}}.$$

PROOF. For $\kappa \in \mathcal{S}'(\mathbf{R}^{j+k})$, define

$$\Xi_{j,k}(\kappa) = \int_{\mathbf{R}^{j+k}} \kappa(s_1, \cdots, s_j; t_1, \cdots, t_k)\partial_{s_1}^* \cdots \partial_{s_j}^* \partial_{t_1} \cdots \partial_{t_k} ds_1 \cdots ds_j dt_1 \cdots dt_k. \tag{3.1.6}$$

Then by Theorem 6.6 in [HKPS]

$$\|\Xi_{j,k}(\kappa)\varphi\|_p \leq \tilde{a}_{\frac{q}{2}} |(A^p)^{\otimes j} \otimes (A^{-(p+q)})^{\otimes k} \kappa|_{L^2(\mathbf{R}^{j+k})} \|\varphi\|_{p+q},$$

where

$$\tilde{a}_{\frac{q}{2}} = sup_{n \geq 0} 2^{-qn} \frac{\sqrt{(j+n)!(k+n)!}}{n!}. \tag{3.1.7}$$

Taking $j = 0$ in (3.1.7), we get

$$\begin{aligned}
\tilde{a}_{\frac{q}{2}} &= sup_{n \geq 0} 2^{-qn} \frac{\sqrt{(n)!(k+n)!}}{n!} \\
&= sup_{n \geq 0} \sqrt{\frac{(k+n)!}{n!} (\frac{1}{2})^{2qn}} \\
&\leq \sqrt{2^q k^k [\frac{2^q}{-2qe(log(\frac{1}{2}))}]^k} \\
&= 2^{\frac{q}{2}} k^{\frac{k}{2}} [\frac{2^q}{2qe(log2)}]^{\frac{k}{2}} = a_{\frac{q}{2}}.
\end{aligned}$$

Thus

$$\begin{aligned}\|\Xi_{0,k}(\kappa)\varphi\|_p &\le \tilde{a}_{\frac{q}{2}}|(A^{-(p+q)})^{\otimes k}\kappa|_{L^2(\mathbf{R}^k)}\|\varphi\|_{p+q}\\ &\le a_{\frac{q}{2}}|\kappa|_{-(p+q)}\|\varphi\|_{p+q}.\end{aligned}$$

Therefore $\Xi_{0,k}(\kappa)$ is continuous from $(\mathcal{S})_{p+q}$ into $(\mathcal{S})_p$ for any $p \ge 0$ and $q > 0$. In particular, it is continuous from $(\mathcal{S})$ into itself. □

REMARK *The operator $\Xi_{j,k}(\kappa)$ defined by (3.1.6) is called an integral kernel operator. See Chapter 10 in [K2] for more information about it.*

LEMMA 3.1.3 *For any $t \in [0,T]$, the operators A_t and E_t are continuous from $(\mathcal{S})$ into itself.*

PROOF. Using κ_t we can rewrite A_t as the following:

$$A_t = \int_{-\infty}^{t} F(t-u)\partial_u du = \int_{-\infty}^{\infty} \kappa_t(u)\partial_u du = \Xi_{0,1}(\kappa_t).$$

Hence by Lemma 3.1.2, the operator A_t is continuous from $(\mathcal{S})$ into itself. Moreover, define

$$\kappa_t^{(n)}(u_1,\cdots,u_n) = F(t-u_1)\cdots F(t-u_n), \quad \text{if} \quad u_1,\cdots,u_n < t; \quad 0, \quad \text{otherwise.} \tag{3.1.8}$$

Then $\kappa_t^{(n)} \in L^2(\mathbf{R}^n)$, and $|\kappa_t^{(n)}|_{-(p+q)} \le |\kappa_t^{(n)}|_0 = |F|_0^n$, for $n \ge 0$. By (3.1.6) we can rewrite A_t^n as

$$\begin{aligned}A_t^n &= \int_{-\infty}^{t} F(t-u_1)\partial_{u_1}du_1\cdots\int_{-\infty}^{t} F(t-u_n)\partial_{u_n}du_n\\ &= \int_{-\infty}^{t}\cdots\int_{-\infty}^{t} F(t-u_1)\cdots F(t-u_n)\partial_{u_1}\cdots\partial_{u_n}du_1\cdots du_n\\ &= \int_{\mathbf{R}^n}\kappa_t^{(n)}(u_1,\cdots,u_n)\partial_{u_1}\cdots\partial_{u_n}du_1\cdots du_n = \Xi_{0,n}(\kappa_t^{(n)}).\end{aligned}$$

Therefore by Lemma 3.1.2, we have for any $\varphi \in (\mathcal{S})_{p+q}$,

$$\|E_t\varphi\|_p \le \sum_{n=0}^{\infty}\frac{\|A_t^n\varphi\|_p}{n!} = \sum_{n=0}^{\infty}\frac{\|\Xi_{0,n}(\kappa_t^{(n)})\varphi\|_p}{n!}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} 2^{\frac{q}{2}} n^{\frac{n}{2}} [\frac{2^q}{2qe(log2)}]^{\frac{n}{2}} |\kappa_t^{(n)}|_{-(p+q)} \|\varphi\|_{p+q} \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} 2^{\frac{q}{2}} n^{\frac{n}{2}} [\frac{2^q}{2qe(log2)}]^{\frac{n}{2}} |F|_0^n \|\varphi\|_{p+q} \\
&\approx \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n} n^n e^{-n}} 2^{\frac{q}{2}} n^{\frac{n}{2}} [\frac{2^q}{2qe(log2)}]^{\frac{n}{2}} |F|_0^n \|\varphi\|_{p+q} \\
&= \|\varphi\|_{p+q} \frac{2^{\frac{q}{2}}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} n^{-\frac{n+1}{2}} (\sqrt{\frac{2^q e}{2q(log2)}} |F|_0)^n \\
&= c_q(F) \|\varphi\|_{p+q},
\end{aligned}
$$

where

$$
c_q(F) = \frac{2^{\frac{q}{2}}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} n^{-\frac{n+1}{2}} (\sqrt{\frac{2^q e}{2q(log2)}} |F|_0)^n < \infty.
$$

This implies that the operator E_t is continuous from $(\mathcal{S})_{p+q}$ into $(\mathcal{S})_p$ for any $p \geq 0$ and $q > 0$. In particular, E_t is continuous from $(\mathcal{S})$ into itself. □

Recall that for any s, $\kappa_s \in L^2(\mathbf{R})$. Therefore

$$
: e^{\langle \cdot, \kappa_s \rangle} := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \kappa_s^{\otimes n} \rangle \in (L^2). \tag{3.1.9}
$$

PROPOSITION 3.1.4 *For any fixed $s \in [0,T]$, the function $E_s^* \Phi, \Phi \in \mathcal{M}^*$ is continuous from $\mathcal{M}^*$ into itself, and*

$$
E_s^* \Phi =: e^{\langle \cdot, \kappa_s \rangle} : \diamond \Phi. \tag{3.1.10}
$$

Moreover, if $\Phi \in (\mathcal{S})^*$, then $E_s^* \Phi \in (\mathcal{S})^*$.

PROOF. For any fixed $s \in [0,T]$, by Lemma 3.1.3 we know that $A_s(: e^{\langle \cdot, \xi \rangle} :) \in (\mathcal{S})$ and $E_s(: e^{\langle \cdot, \xi \rangle} :) \in (\mathcal{S})$ for any $\xi \in \mathcal{S}_c(\mathbf{R})$. Therefore for any $\Phi \in (\mathcal{S})^*$,

$$
\begin{aligned}
&\ll \Phi, A_s(: e^{\langle \cdot, \xi \rangle} :) \gg = \ll A_s^* \Phi, : e^{\langle \cdot, \xi \rangle} : \gg \\
&= S(A_s^* \Phi)(\xi) = \langle \kappa_s, \xi \rangle S\Phi(\xi) \\
&= \langle \kappa_s, \xi \rangle \ll \Phi, : e^{\langle \cdot, \xi \rangle} : \gg = \ll \Phi, \langle \kappa_s, \xi \rangle : e^{\langle \cdot, \xi \rangle} : \gg .
\end{aligned}
$$

This implies that

$$
A_s(: e^{\langle \cdot, \xi \rangle} :) = \langle \kappa_s, \xi \rangle : e^{\langle \cdot, \xi \rangle} : . \tag{3.1.11}
$$

By induction we have for $n \geq 0$:

$$A^n(:e^{\langle\cdot,\xi\rangle}:) = (\langle\kappa_s,\xi\rangle)^n : e^{\langle\cdot,\xi\rangle} : . \tag{3.1.12}$$

Therefore

$$\begin{aligned} E_s(:e^{\langle\cdot,\xi\rangle}:) &= \sum_{n=0}^{\infty} \frac{A^n}{n!} : e^{\langle\cdot,\xi\rangle} : \\ &= \sum_{n=0}^{\infty} \frac{(\langle\kappa_s,\xi\rangle)^n}{n!} : e^{\langle\cdot,\xi\rangle} := e^{\langle\kappa_s,\xi\rangle} : e^{\langle\cdot,\xi\rangle} : . \end{aligned}$$

So we have

$$\begin{aligned} S(E_s^*\Phi)(\xi) &= \langle\!\langle E_s^*\Phi, : e^{\langle\cdot,\xi\rangle} : \rangle\!\rangle = \langle\!\langle \Phi, E_s : e^{\langle\cdot,\xi\rangle} : \rangle\!\rangle \\ &= e^{\langle\kappa_s,\xi\rangle} S\Phi(\xi) = S(: e^{\langle\cdot,\xi\rangle} : \diamond\Phi)(\xi). \end{aligned}$$

From this and the injectivity of S-transform, we get (3.1.10). Therefore $E_s^*\Phi, \Phi \in \mathcal{M}^*$ is continuous from $\mathcal{M}^*$ into itself. If $\Phi \in (\mathcal{S})^*$, then (3.1.10) implies that $E_s^*\Phi \in (\mathcal{S})^*$. Thus the proposition is proved. □

THEOREM 3.1.5 *Let $f(s,\Phi) = A_s^*\Phi : [0,T] \times (\mathcal{S})^* \to (\mathcal{S})^*$ and $\theta \in \mathcal{A}_f$. Then equation (3.1.1) has a unique solution $X : [0,T] \to (\mathcal{S})^*$ given by*

$$X_t = \theta_t + \int_0^t \theta_s \diamond \langle\cdot,\kappa_s\rangle \diamond \Phi_{s,t} ds, \quad t \in [0,T], \tag{3.1.13}$$

where $\Phi_{s,t}$ is in $(\mathcal{S})^*$ whose S-transform is given by $e^{\int_s^t \langle\kappa_u,\xi\rangle du}$.

Let $g(s,\Phi) = E_s^*\Phi : [0,T] \times \mathcal{M}^* \to \mathcal{M}^*$ and $\theta \in \mathcal{B}_g$. Then equation (3.1.2) has a unique solution $Y : [0,T] \to \mathcal{M}^*$ given by

$$X_t = \theta_t + \int_0^t \theta_s \diamond : e^{\langle\cdot,\kappa_s\rangle} : \diamond\Psi_{s,t} ds, \quad t \in [0,T], \tag{3.1.14}$$

where $\Psi_{s,t}$ is in $\mathcal{M}^*$ whose S-transform is given by $\exp[\int_s^t e^{\langle\kappa_u,\xi\rangle} du]$.

PROOF. By Proposition 3.1.1 and Proposition 3.1.4, f satisfies conditions (A1-A3) for $\beta = 0$, and g satisfies conditions (F1-F3). Thus there exists a unique solution for each equation by Theorem 1.3.2 and Theorem 2.3.2, respectively. Therefore we only need to verify (3.1.13) and (3.1.14).

Denote $F_t(\xi) = SX_t(\xi)$, and $g_t(\xi) = S\theta_t(\xi)$. Takeing S-transform on both sides of equation (3.1.1), we have

$$F_t(\xi) = g_t(\xi) + \int_0^t \langle \kappa_s, \xi \rangle F_s(\xi) ds, \quad t \in [0,T],$$

And its solution can be obtained following a computation similar to that in Example 1.4.3, i.e.

$$F_t\xi) = g_t(\xi) + \int_0^t g_s(\xi)\langle \kappa_s, \xi \rangle e^{\int_s^t \langle \kappa_u, \xi \rangle du} ds, \quad t \in [0,T].$$

Since S-transform is injective, this implies that

$$X_t = \theta_t + \int_0^t \theta_s \diamond \langle \cdot, \kappa_s \rangle \diamond \Phi_{s,t} ds, \quad t \in [0,T],$$

where $\Phi_{s,t}$ is the Hida distribution in $(S)^*$ whose S-transform is given by $e^{\int_s^t \langle \kappa_u, \xi \rangle du}$, and the latter is a U-functional since

$$|e^{\int_s^t \langle \kappa_u, \xi \rangle du}| \leq e^{T|F|_0 |\xi|_p} \leq e^{1+T^2|F|_0^2|\xi|_p^2}. \tag{3.1.15}$$

This proves (3.1.13). On the other hand, taking S-transform on both sides of (3.1.2), and using Proposition 3.1.4 we get

$$F_t(\xi) = g_t(\xi) + \int_0^t e^{\langle \kappa_s, \xi \rangle} F_s(\xi) ds. \tag{3.1.16}$$

Solving this equation and then taking the inverse S-transform gives us (3.1.14), and so the theorem is proved. □

The following theorem strengthens the first result in Theorem 3.1.5.

THEOREM 3.1.6 *Let $F \in L^2(\mathbf{R}^+)$, $f = A_s^* \Phi$ and $\theta \in \mathcal{A}_f$ be as in Theorem 3.1.5. Suppose there exist constants $r > 0$, $a \geq 0$, and a function $K \in L^1[0,T]$, such that for almost all $t \in [0,T]$, and all $\xi \in S_c(\mathbf{R})$,*

$$|S\theta_t(\xi)| \leq K(t) \exp[a|\xi|_{-r}^2]. \tag{3.1.17}$$

Moreover, suppose there exists $q \geq 0$ satisfying $2(a+1)e^2\|A^{-(q\wedge r)}\|_{HS}^2 < 1$, such that

$$|F|_{j,k} \equiv |x^j F^{(k)}(x)|_{L^2(\mathbf{R}^+)} < \infty, \quad \forall j,k = 0,1,\cdots,2q. \tag{3.1.18}$$

Then equation (3.1.1) has a unique solution $X : [0,T] \to (L^2)$.

PROOF. By Theorem 3.1.5, equation (3.1.1)has a unique solution given by (3.1.13). Thus

$$SX_t(\xi) = S\theta_t(\xi) + \int_0^t S\theta_s(\xi)\langle\kappa_s,\xi\rangle \exp[\int_s^t \langle\kappa_u,\xi\rangle du]ds, \quad t \in [0,T]. \tag{3.1.19}$$

From (3.1.18) and the definition of $|\cdot|_q, q \geq 0$, we have for $s \in [0,T]$,

$$\begin{aligned}
|\kappa_s|_1 &= |A\kappa_s|_0 = |-\kappa_s''(\cdot) + (\cdot^2+1)\kappa_s(\cdot)|_0 \\
&= |-F''(s-\cdot) + (\cdot^2+1)F(s-\cdot)|_{L^2(-\infty,s]} \\
&= |-F''(\cdot) + (s^2 - 2s\cdot + \cdot^2 + 1)F(\cdot)|_{L^2(\mathbf{R}^+)} \\
&\leq |F|_{0,2} + |F|_{2,0} + 2s|F|_{1,0} + (s^2+1)|F|_{0,0} \\
&\leq |F|_{0,2} + |F|_{2,0} + 2T|F|_{1,0} + (T^2+1)|F|_{0,0}.
\end{aligned}$$

Similarly

$$|\kappa_s|_q \leq \sum_{n=0}^{2q} a_n T^n \equiv C < \infty, \quad \forall s \in [0,T],$$

where a_n is a linear combination of $\{|F|_{j,k}, j,k = 0,1,\cdots,2q\}$. Therefore, for all $s,t \in [0,T]$, and $\xi \in \mathcal{S}_c(\mathbf{R})$,

$$\begin{aligned}
&|\exp[\int_s^t \langle\kappa_u,\xi\rangle du]| \leq \exp[\int_s^t |\langle\kappa_u,\xi\rangle| du] \\
&\leq \exp[\int_s^t |\kappa_u|_q |\xi|_{-q} du] \leq \exp[TC|\xi|_{-q}].
\end{aligned}$$

Combinig (3.1.17), (3.1.19) and (3.1.20) we have for $t \in [0,T]$,

$$|SX_t(\xi)| \leq (K(t) + (Ce^{1+C^2}|K|_{L^1}))\exp[(a+T)|\xi|^2_{-(q\wedge r)}].$$

This and Theorem 1.1.5 imply that $X_t \in (L^2)$ for $t \in [0,T]$. $\square$

EXAMPLE 3.1.9 Let $F(t) = e^{-t}(t \geq 0)$, and $\theta_t =: e^{\langle \cdot, \eta \rangle} : (t \in [0,T])$, for $\eta \in \mathcal{S}_c(\mathbf{R})$. Then $F(t)$ satisfies (3.1.18) for all $q \geq 0$. Moreover, for any $r \geq 0$,

$$|S\theta_t(\xi)| = |e^{\langle \eta, \xi \rangle}| \leq \exp[|\eta|_r |\xi|_{-r}] \leq \exp[\frac{|\eta|_r^2}{2}] \exp[\frac{|\xi|_{-r}^2}{2}].$$

Therefore, by taking $a = \frac{1}{2}$, $K(t) = \exp[\frac{|\eta|_r^2}{2}]$, and r, q sufficiently large in Theorem 3.1.6, we conclude that the following equation

$$X_t = \theta_t + \int_0^t A_s^* X_s ds, \quad t \in [0,T],$$

has a unique solution $X \in (L^2)$ given by

$$X_t = \theta_t + \int_0^t \theta_s \diamond \langle \cdot, \kappa_s \rangle \diamond : e^{\int_s^t \langle \cdot, \kappa_u \rangle du} : ds.$$

3.2. A Linear Volterra Equation

In this section we study the following white noise Volterra equation

$$X_t = \theta_t + \int_0^t \sigma(t,s) \Phi_s \diamond X_s ds, \quad t \in [0,T]. \tag{3.2.1}$$

Here $\theta, \Phi : [0,T] \to (\mathcal{S})^*$ are generalized functions, and $\sigma : [0,T]^2 \to \mathbf{R}$ is a deterministic function. For a function $f \in L^2([0,T])$ or $L^2([0,T]^2)$, denote by $|f|_0$ its L^2-norm. Assume that (θ, Φ, σ) satisfy the following conditions:

(E1) (Measurability Condition) The functions $s \mapsto S\theta_s(\xi), s \mapsto S\Phi_s(\xi)$ are measurable on [0, T] for all $\xi \in \mathcal{S}_c(\mathbf{R})$.

(E2) (Growth Condition) There exist nonnegative numbers a and p, and a function $J \in L^4[0,T]$, such that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|S\Phi_s(\xi)|, |S\theta_s(\xi)| \leq J(s) \exp[a|\xi|_p^{\frac{2}{1-\beta}}].$$

(E3) Both σ and σJ are in $L^2([0,T]^2)$, and $\sigma(t,s) = 0$, for $0 \leq t < s$.

LEMMA 3.2.1 *Suppose Φ and σ satisfy the above conditions (E1)-(E3). For $s,t \in [0,T]$ and $n = 1, 2, \cdots$, define*

$$K_1(t,s) = \sigma(t,s)\Phi_s, \quad K_{n+1}(t,s) = \int_0^t K_1(t,u) \diamond K_n(u,s) du. \tag{3.2.2}$$

Then all the functionals $u \mapsto K_1(t,u) \diamond K_n(u,s)$ are Pettis integrable in $(S)^*$.

PROOF. For any fixed $s, t \in [0,T]$, we show by induction that the function $u \mapsto K_1(t,u) \diamond K_n(u,s)$ is Pettis integrable on [0, T] for $n \geq 1$. When $n = 1$, the weakly measurability of $K_1(t,u) \diamond K_1(u,s), u \in [0,T]$, is obvious since

$$S(K_1(t,u) \diamond K_1(u,s))(\xi) = \sigma(t,u) S\Phi_u(\xi) \sigma(u,s) S\Phi_s(\xi). \tag{3.2.3}$$

Furthermore, for all $\xi \in \mathcal{S}_c(\mathbf{R})$,

$$\begin{aligned} &\int_0^T |S(K_1(t,u) \diamond K_1(u,s))(\xi)| du \\ \leq\ & |S\Phi_s(\xi)| \exp[a|\xi|_p^{\frac{2}{1-\beta}}] \int_0^T |\sigma(t,u) J(u) \sigma(u,s)| du \\ \leq\ & J(s) \sqrt{\int_0^T \sigma^2(t,u) du} \sqrt{\int_0^T J^2(u) \sigma^2(u,s) du}\, \exp[2a|\xi|_p^{\frac{2}{1-\beta}}]. \end{aligned}$$

Thus $K_1(t,u) \diamond K_1(u,s), u \in [0,T]$, satisfies condition $(G2)$. Therefore by Lemma 2.2.2, $K_1(t,u) \diamond K_1(u,s), u \in [0,T]$, is Pettis integrable.

Suppose each $K_1(t,u) \diamond K_n(u,s), u \in [0,T]$, is Pettis integrable for $n \leq k$. Denote

$$K_{k+1}(t,s) = \int_0^t K_1(t,u) \diamond K_k(u,s) du. \tag{3.2.4}$$

Then it is easily seen that the function $u \mapsto K_1(t,u) \diamond K_{k+1}(u,s)$, is weakly measurable on $[0,T]$. Furthermore, we have for all $\xi \in \mathcal{S}_c(\mathbf{R})$,

$$\begin{aligned} \textstyle\int_0^t |S(K_1(t,u) \diamond K_{k+1}(u,s))(\xi)| du \leq J(s) \exp[(k+2)a|\xi|_p^{\frac{2}{1-\beta}}] \\ \{\textstyle\int_0^t |\sigma(t,u_1) J(u_1)| du_1 \int_0^{u_1} |\sigma(u_1,u_2) J(u_2)| du_2 \\ \cdots \textstyle\int_0^{u_k} |\sigma(u_k,u_{k+1}) J(u_{k+1}) \sigma(u_{k+1},s)| du_{k+1}\}. \end{aligned}$$

Therefore the function $K_1(t,u) \diamond K_{k+1}(u,s), u \in [0,T]$, satisfies condition (G2). By Lemma 2.2.2, $K_1(t,u) \diamond K_{k+1}(u,s), u \in [0,T]$, is Pettis integrable. Thus by induction, we showed that for $n = 1, 2, \cdots$, each integral $\int_0^t K_1(t,u) \diamond K_n(u,s) du$ exists in $(S)^*$ in the Pettis sense. □

LEMMA 3.2.2 *Suppose Φ and σ satisfy conditions (E1)-(E3). For $s,t \in [0,T]$ and $n = 1,2,\cdots$, let $K_n(t,s)$ be defined as in Lemma 3.2.1. Define*

$$H(t,s) = \sum_{n=1}^{\infty} K_n(t,s). \tag{3.2.5}$$

Then $\sum_{n=1}^{\infty} \ll K_n(t,s), \varphi \gg$ converges for all $\varphi \in \mathcal{M}$, and $H(t,s) \in \mathcal{M}^*$.

PROOF. Since $|S\Phi_u(\xi)| \leq J_u \exp[a|\xi|_p^{\frac{2}{1-\beta}}]$ and by (E3), the function $u \mapsto \sigma(t,u)S\Phi_u(\xi)$ is in $L^2[0,T]$ for all $t \in [0,T]$ and $\xi \in S_c(\mathbf{R})$. Set

$$A^2(t) = \int_0^T |\sigma(t,u)S\Phi_u(\xi)|^2 du, \tag{3.2.6}$$

$$B^2(s) = |S\Phi_s(\xi)|^2 \int_0^T |\sigma(u,s)|^2 du, \tag{3.2.7}$$

and denote $b = |\sigma J|_0^2$. Then it is easily seen that

$$\int_0^t A^2(u)du \leq b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]. \tag{3.2.8}$$

Moreover, by induction we can check

$$|SK_{n+2}(t,s)(\xi)|^2 \leq A^2(t)B^2(s)\frac{(\int_0^t A^2(u)du)^n}{n!}.$$

Therefore

$$\begin{aligned}
& \sum_{n=0}^{\infty} |SK_{n+2}(t,s)(\xi)| \\
\leq\ & \sum_{n=0}^{\infty} A(t)B(s)\frac{(\int_0^t A^2(u)du)^{\frac{n}{2}}}{\sqrt{n!}} \\
\leq\ & A(t)B(s)\sum_{n=0}^{\infty} \frac{(b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}])^{\frac{n}{2}}}{\sqrt{n!}} \quad (\text{by}(3.2.8)) \\
\leq\ & A(t)B(s)\sqrt{2}\exp[\int_0^t A^2(u)du] \\
\leq\ & \exp[a|\xi|_p^{\frac{2}{1-\beta}}]\sqrt{\int_0^t \sigma(t,u)^2 J^2(u)du} \cdot J(s)\exp[a|\xi|_p^{\frac{2}{1-\beta}}]\sqrt{\int_0^t \sigma(u,s)^2 du} \\
& \cdot\sqrt{2}\exp[\int_0^t A^2(u)du] \\
\leq\ & C(s,t)\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]],
\end{aligned}$$

where

$$C(s,t) = \sqrt{2}J(s)\sqrt{\int_0^T \sigma(t,u)^2 J^2(u)du}\sqrt{\int_0^T \sigma(u,s)^2 du}$$

is integrable on $[0,T]^2$ by conditions (E2) and (E3).Thus for any fixed $\xi, \eta \in \mathcal{S}_c(\mathbf{R})$, we have,

$$\sum_{n=0}^{\infty} |SK_{n+2}(t,s)(z\xi+\eta)| \le A(t)B(s)\sum_{n=0}^{\infty} \frac{(b\exp[2a(|z|^2|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^2)])^{\frac{n}{2}}}{\sqrt{n!}}. \quad (3.2.9)$$

This implies that the series $\sum_{n=0}^{\infty} |SK_n(t,s)(z\xi+\eta)|$ converges uniformly on any compact subset of $\mathbf{C}$ and on $[0,T]^2$. Thus $z \mapsto \sum_{n=0}^{\infty} SK_n(t,s)(z\xi+\eta)$ is entire on $\mathbf{C}$. Therefore the function $\xi \mapsto \sum_{n=0}^{\infty} SK_n(t,s)(\xi)$ is a quasi-U-functional. Thus there exists a unique distribution in $\mathcal{M}^*$, denoted by $H(t,s) = \sum_{n=1}^{\infty} K_n(t,s)$, such that

$$SH(t,s)(\xi) = \sum_{n=1}^{\infty} SK_n(t,s)(\xi), \quad \xi \in \mathcal{S}_c(\mathbf{R}). \quad (3.2.10)$$

By a limiting argument and (3.2.9), the series $\sum_{n=0}^{\infty} \ll K_n, \varphi \gg$ converges for each $\varphi \in \mathcal{M}$. Thus the lemma is proved. □

THEOREM 3.2.3 *Suppose θ, Φ and σ satisfy conditions (E1)-(E3). For $s,t \in [0,T]$ and $n = 1,2,\cdots$, let $K_n(t,s)$ and $H(t,s)$ be defined as in Lemmas 3.2.1 and 3.2.2. Then for $t \in [0,T]$, the white noise integral $\int_0^t H(t,s) \diamond \theta_s ds$ exists in $\mathcal{M}^*$ in the Pettis sense, and equation (3.2.1) has a unique solution in $\mathcal{M}^*$ given by*

$$X_t = \theta_t + \int_0^t H(t,s) \diamond \theta_s ds, \quad t \in [0,T]. \quad (3.2.11)$$

PROOF. From the argument of Lemma 3.2.2 we see that the function $s \mapsto S(H(t,s) \diamond \theta_s)(\xi)$ is measurable for each $t \in [0,T]$ and $\xi \in \mathcal{S}_c(\mathbf{R})$. Thus $H(t,s) \diamond \theta_s, s \in [0,T]$, satisfies condition $(G1)$. Furthermore, for any $\xi \in \mathcal{S}_c(\mathbf{R})$, we have

$$\begin{aligned} &\int_0^t |SH(t,s)(\xi) S\theta_s(\xi)| ds \\ \le\ &\int_0^t C(s,t)J(s)ds \cdot \exp[3a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]] \\ \le\ & f(t)\exp[3a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]], \end{aligned}$$

where

$$\begin{aligned} f(t) &= \int_0^t C(s,t)J(s)ds \\ &= \sqrt{2}\int_0^t J^2(s)\sqrt{\int_0^T \sigma(t,u)^2 J^2(u)du}\sqrt{\int_0^T \sigma(u,s)^2 duds} \\ &\leq \sqrt{2}\sqrt{\int_0^T J^4(s)ds}\sqrt{\int_0^T \sigma(t,u)^2 J^2(u)du}\sqrt{\int_0^T ds\int_0^T \sigma(u,s)^2 du} \\ &\in L^2[0,T] \quad (\text{by}(E2-E3)). \end{aligned}$$

Therefore $H(t,s)\diamond\theta_s, s \in [0,T]$, satisfies condition $(G2)$. Thus $H(t,s)\diamond\theta_s, s \in [0,T]$, is Pettis integrable by Lemma 2.2.2.

For fixed $\xi \in \mathcal{S}_c(\mathbf{R})$, consider the following deterministic equation:

$$SX_t(\xi) = S\theta_t(\xi) + \int_0^t \sigma(t,s)S\Phi_s(\xi)SX_s(\xi)ds, \quad t \in [0,T]. \tag{3.2.12}$$

By Theorem 1.5 in [T], its unique solution is given by

$$SX_t(\xi) = S\theta_t(\xi) + \int_0^t SH(t,s)(\xi)S\theta_s(\xi)ds, \quad t \in [0,T], \tag{3.2.13}$$

where $SH(t,s)(\xi) = \sum_{n=0}^{\infty} SK_n(t,s)(\xi)$. This implies that

$$SX_t(\xi) = S(\theta_t + \int_0^t H(t,s)\diamond\theta_s ds)(\xi). \tag{3.2.14}$$

Therefore we get (3.2.11) from the injectivity of S-transform.

Now we show that X given by (3.2.11) is a solution of equation (3.2.1). It is obvious that X is weakly measurable. To check the integrability condition in Definition 2.3.1, we need to show that the function $\sigma(t,s)\Phi_s \diamond X_s, s \in [0,T]$, is Pettis integrable. It is obviously weakly measurable. Moreover, since

$$\begin{aligned} |SX_t(\xi)| &\leq |S\theta_t(\xi)| + \int_0^t |SH(t,s)(\xi)S\theta_s(\xi)|ds \\ &\leq J(t)\exp[a|\xi|_p^{\frac{2}{1-\beta}}] + f(t)\exp[3a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]], \end{aligned}$$

we have

$$\begin{aligned}
&\int_0^t |\sigma(t,s)S\Phi_s(\xi)SX_s(\xi)|ds \\
\leq\ & \exp[2a|\xi|_p^{\frac{2}{1-\beta}}]\int_0^t |\sigma(t,s)|J^2(s)ds \\
+\ & \exp[4a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]]\int_0^t |\sigma(t,s)|J(s)f(s)ds \\
\leq\ & \exp[4a|\xi|_p^{\frac{2}{1-\beta}}]\exp[b\exp[2a|\xi|_p^{\frac{2}{1-\beta}}]](|\sigma(t,\cdot)|_0|J^2|_0 + |\sigma(t,\cdot)J(\cdot)|_0|f|_0).
\end{aligned}$$

Thus by conditions (E2-E3) and (3.2.12), the function $\sigma(t,s)\Phi_s \diamond X_s, s \in [0,T]$, satisfies condition (G2). By Lemma 2.2.2, it is Pettis integrable, i.e. X satisfies the integrability condition in Definition 2.3.1. Furthermore, since $SX_t(\xi)$ is the unique solution of (3.2.12), we have

$$\begin{aligned}
SX_t(\xi) &= S\theta_t(\xi) + \int_0^t \sigma(t,s)S\Phi_s(\xi)SX_s(\xi)ds \\
&= S(\theta_t + \int_0^t \sigma(t,s)\Phi_s \diamond X_s ds)(\xi).
\end{aligned}$$

This implies that

$$X_t = \theta_t + \int_0^t \sigma(t,s)\Phi_s \diamond X_s ds,$$

i.e., for any $\varphi \in \mathcal{M}$

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \ll \int_0^t \sigma(t,s)\Phi_s \diamond X_s ds, \varphi \gg .$$

By this and the definition of Pettis integrable, we have for any $\varphi \in \mathcal{M}$

$$\ll X_t, \varphi \gg = \ll \theta_t, \varphi \gg + \int_0^t \ll \sigma(t,s)\Phi_s \diamond X_s, \varphi \gg ds.$$

Thus condition (3) in Definition 2.3.1 is satisfied, and so X defined by (3.2.11) is a solution of (3.2.1). The uniqueness follows from the uniqueness of the deterministic equation (3.2.12). Thus the proof is finished. □

Following an argument similar to that in the above proof, we can strengthen the result in this theorem under stronger growth conditions. That is, we replace condition $(E2)$ by the following conditions $(E2)'$ or $(E2)''$:

$(E2)'$ There exist a nonnegative function $J \in L^4[0,T]$, and nonnegative numbers a and p, such that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|S\Phi_s| \leq J(s)|\xi|_p, \quad |S\theta_s| \leq J(s)\exp[a|\xi|_p^{\frac{2}{1-\beta}}].$$

$(E2)''$ There exist a nonnegative function $J \in L^4[0,T]$, and nonnegative numbers a and p, with $2(1+a+b)e^2\|A^{-p}\|_{HS}^2 < 1$, where $b = |\sigma J|_0^2$, such that for all $\xi \in \mathcal{S}_c(\mathbf{R})$, and almost all $s \in [0,T]$,

$$|S\Phi_s| \leq J(s)|\xi|_{-p}, \quad |S\theta_s| \leq J(s)e^{a|\xi|_{-p}^2}.$$

Then we get the following inequalities instead of (3.2.8):

$$\int_0^t A^2(u)du \leq b|\xi|_p^2, \tag{3.2.8$'$}$$

$$\int_0^t A^2(u)du \leq b|\xi|_{-p}^2. \tag{3.2.8$''$}$$

Thus we have

$$\sum_{n=1}^{\infty} |SK_n(s,t)| \leq C(s,t)e^{(1+b)|\xi|_p^{\frac{2}{1-\beta}}},$$

$$\sum_{n=1}^{\infty} |SK_n(s,t)| \leq C(s,t)e^{(1+b)|\xi|_{-p}^2}.$$

Hence under condition $(E2)'$,

$$\begin{aligned}|SX_t(\xi)| &\leq J(t)\exp[a|\xi|_p^{\frac{2}{1-\beta}}] + f(t)\exp[(a+1+b)|\xi|_p^{\frac{2}{1-\beta}}] \\ &\leq (J(t)+f(t))\exp[(a+1+b)|\xi|_p^{\frac{2}{1-\beta}}].\end{aligned}$$

This implies that $X \in (\mathcal{S})_\beta^*$ by Theorem 1.1.3.

On the other hand, under condition $(E2)''$,

$$\begin{aligned}|SX_t(\xi)| &\leq J(t)e^{a|\xi|_{-p}^2} + f(t)e^{(a+1+b)|\xi|_{-p}^2} \\ &\leq (J(t)+f(t))e^{(a+1+b)|\xi|_{-p}^2}.\end{aligned}$$

Therefore $X \in (L^2)$ by Theorem 1.1.5. Summing up we get

COROLLARY 3.2.4

(1) If θ, Φ and σ satisfy conditions $(E1), (E2)'$ and $(E3)$, then equation (3.2.1) has a unique solution $X : [0,T] \to (\mathcal{S})^*_\beta$ given by (3.2.2), (3.2.5) and (3.2.11).

(2) If θ, Φ and σ satisfy conditions $(E1), (E2)''$ and (E3), then equation (3.2.1) has a unique solution $X : [0,T] \to (L^2)$ given by (3.2.2), (3.2.5) and (3.2.11).

EXAMPLE 3.2.5 Let $\sigma(t,s) = (t-s)^{-\frac{1}{4}}$, if $s < t$; 0, if $s \geq t$. It is easy to check that the equation

$$X_t =: e^{\dot{B}(t)} : + \int_0^t \sigma(t,s)\dot{B}(s) \diamond X_s ds, \quad t \in [0,T],$$

has a unique weak solution $X : [0,T] \to (\mathcal{S})^*_\beta$.

EXAMPLE 3.2.6 Let σ be bounded and satisfy (E3). Let $\Phi_s = \dot{B}(s)$, then by (1) in the above corollary, (3.2.1) has a unique solution in $(\mathcal{S})^*$. This gives us the result in [ØZ].

3.3. A Partial Differential Equation

Consider the following white noise partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u + : e^{\dot{B}_x} : \diamond u, \quad u(0,x) = f(x), \quad x \in \mathbf{R}^n, \quad t \in [0,\infty). \tag{3.3.1}$$

Here $f(x)$ is a deterministic function, with bounded derivative on $\mathbf{R}^n$, $\dot{B}_x = \langle \cdot, \delta_x \rangle$ is the white noise, and

$$: e^{\dot{B}_x} := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \delta_x^{\otimes n} \rangle \in (\mathcal{S})^*. \tag{3.3.2}$$

If $\Phi \in (\mathcal{S})^*$, then it is easily seen that the function $e^{S\Phi(\xi)}, \xi \in \mathcal{S}_c(\mathbf{R}^n)$, is a quasi-U-functional.

DEFINITION 3.3.1 *For any $\Phi \in (\mathcal{S})^*$, define $: e^\Phi : \in \mathcal{M}^*$ to be the unique distribution whose S-transform is given by $S(: e^\Phi :)(\xi) = e^{S\Phi(\xi)}, \xi \in \mathcal{S}_c(\mathbf{R}^n)$.*

LEMMA 3.3.2 *Let b_t be the standard Brownian motion in $\mathbf{R}^n$ defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and E_x be the expectation with respect to P_x. Define for $t \in \mathbf{R}^+$, $x \in \mathbf{R}^n$ and $\xi \in \mathcal{S}_c(\mathbf{R}^n)$*

$$F(t,x)(\xi) = E_x(f(b_t)\exp[\int_0^t e^{\xi(b_s)}ds]). \tag{3.3.3}$$

Then $F(t,x)(\xi), \xi \in \mathcal{S}_c(\mathbf{R}^n)$, is a quasi-U-functional for all $t \geq 0$ and $x \in \mathbf{R}^n$.

PROOF. Pick and fix any $t \geq 0$ and $x \in \mathbf{R}^n$. Denote $F(\xi) = F(t,x)(\xi)$ for simplicity. Notice that there exists $p \geq 0$ such that $|\delta_x|_{-p} < 1$ for all $x \in \mathbf{R}^n$. Let C be the upper bound of $f(x), x \in \mathbf{R}^n$. Then for all $z \in \mathbf{C}$, and all $\xi \in \mathcal{S}_c(\mathbf{R}^n)$,

$$\begin{aligned} &|F(z\xi)| \leq CE_x(\exp[\int_0^t e^{|z||\xi(b_s)|}ds]) \\ \leq\ & CE_x(\exp[\int_0^t e^{|z||\xi|_p|\delta_{b_s}|_{-p}}ds]) \leq C\exp[Te^{|z||\xi|_p}]. \end{aligned}$$

Therefore for all $R \geq 0$, and all $\xi \in \mathcal{S}_c(\mathbf{R}^n)$, with $|\xi|_p \leq 1$,

$$M(R,\xi) \equiv \sup_{|z|=R, z\in\mathbf{C}} |F(z\xi)| \leq C\exp[Te^R].$$

On the other hand, for any fixed $\xi, \eta \in \mathcal{S}_c(\mathbf{R}^n)$,

$$\exp[\int_0^t e^{z\xi(b_s)+\eta(b_s)}ds] = \sum_{n=0}^{\infty}\frac{1}{n!}(\int_0^t e^{z\xi(b_s)+\eta(b_s)}ds)^n,$$

and the above series converges uniformly on any bounded subset of $\mathbf{C}$. Moreover, each function $z \mapsto (\int_0^t e^{z\xi(b_s)+\eta(b_s)}ds)^n$ is entire on $\mathbf{C}$ for $n \geq 0$. Thus the function

$$f(b_t)\exp[\int_0^t e^{z\xi(b_s)+\eta(b_s)}ds], \quad z \in \mathbf{C},$$

is entire. Furthermore, we have

$$|f(b_t)\exp[\int_0^t e^{z\xi(b_s)+\eta(b_s)}ds]| \leq C\exp[te^{|z||\xi|_p+|\eta|_p}]. \tag{3.3.4}$$

Thus for $\xi, \eta \in \mathcal{S}_c(\mathbf{R}^n)$, the function $z \mapsto F(z\xi+\eta)$, is entire on $\mathbf{C}$. Therefore $F(\xi), \xi \in \mathcal{S}_c(\mathbf{R}^n)$, is a quasi-U-functional and Lemma 3.3.2 is proved. $\square$

By Lemma 3.3.2 and the characterization theorem in $\mathcal{M}^*$, there exists a unique distribution in $\mathcal{M}^*$, denoted by $u(t,x)$, such that

$$Su(t,x)(\xi) = F(t,x)(\xi) = E_x(f(b_t)\exp[\int_0^t e^{\xi(b_s)}ds]). \tag{3.3.5}$$

Since $\int_0^t : e^{\dot{B}_{b_s}} : ds \in (\mathcal{S})^*$, we have by definition that $: e^{\int_0^t :e^{\dot{B}_{b_s}}:ds} :\in \mathcal{M}^*$ is the distribution whose S-transform is given by $\exp[\int_0^t e^{\xi(b_s)}ds]$. Define

$$u(t,x) = E_x(f(b_t) : e^{\int_0^t :e^{\dot{B}_{b_s}}:ds} :) \tag{3.3.6}$$

to be the unique Myere-Yan distribution whose S-transform is given by (3.3.5).

LEMMA 3.3.3 *Let $F(t,x)(\xi)$ be defined by (3.3.5). Then, for $t \in \mathbf{R}^+$, $x \in \mathbf{R}^n$ and $\xi \in \mathcal{S}_c(\mathbf{R}^n)$, the derivative $\frac{\partial F}{\partial t}(t,x)(\xi)$ exists, and the function $\frac{\partial F}{\partial t}(t,x)(\xi), \xi \in \mathcal{S}_c(\mathbf{R}^n)$, is a quasi-U-functional.*

PROOF. Set

$$\begin{aligned} g(b_{u_1},\cdots,b_{u_k})(\xi) &= \exp[\xi(b_{u_1})+\cdots+\xi(b_{u_k})], \\ v_k(t,x)(\xi) &= E_x(f(b_t)(\int_0^t e^{\xi(b_s)}ds)^k), \\ F(t,x)(\xi) &= \sum_{k=0}^{\infty}\frac{1}{k!}v_k(t,x)(\xi). \end{aligned}$$

Using Fubini Theorem to rewrite $v_k(t,x)(\xi)$ as the following:

$$\begin{aligned} v_k(t,x)(\xi) &= E_x(f(b_t)\int_0^t\cdots\int_0^t g(b_{u_1},\cdots,b_{u_k})(\xi)du_1\cdots du_k) \\ &= k!E_x(f(b_t)\int_0^t\int_0^{u_k}\cdots\int_0^{u_2} g(b_{u_1},\cdots,b_{u_k})du_1\cdots du_k) \\ &= k!\int_0^t\int_0^{u_k}\cdots\int_0^{u_2} E_x(f(b_t)g(b_{u_1},\cdots,b_{u_k})(\xi))du_1\cdots du_k. \end{aligned}$$

Let

$$p_t(x) = (2\pi t)^{-\frac{n}{2}}\exp[-\frac{|x|^2}{2t}]$$

be the transform distribution of the Brownian motion b_t. Let

$$\mathcal{F}_u = \sigma\{b_s : s \le u\}$$

be the σ-field generated by all $b_s, s \leq u$. Then for any $x \in \mathbf{R}^n$

$$\begin{aligned}
& |E_x((\frac{|b_{t-u_k} - b_0|^2}{(t-u_k)^2} - \frac{n}{t-u_k})f(b_{t-u_k}))| \\
= \; & |E_x((\frac{|b_{t-u_k} - b_0|^2}{(t-u_k)^2} - \frac{n}{t-u_k})(f(b_{t-u_k}) - f(b_0)))| \\
\leq \; & \frac{1}{(t-u_k)^2}\sqrt{E_x((|b_{t-u_k} - b_0|^2 - n(t-u_k))^2)} \\
& \quad \cdot\sqrt{E_x((f(b_{t-u_k}) - f(b_0)))^2} \leq \frac{C_1}{\sqrt{t-u_k}}
\end{aligned}$$

where C_1 is a constant. On the other hand, by the Markov property of Brownian motion, we have

$$\begin{aligned}
& \frac{\partial}{\partial t}E_x(f(b_t)g(b_{u_1}, \cdots, b_{u_k})(\xi)) \\
= \; & \frac{\partial}{\partial t}(\int_{\mathbf{R}^{k+1}} g(x_1, \cdots, x_k)f(x_{k+1})p_{u_1}(x_1 - x)p_{u_2-u_1}(x_2 - x_1)\cdots \\
& \cdots p_{u_k-u_{k-1}}(x_k - x_{k-1})p_{t-u_k}(x_{k+1} - x_k)dx_1 \cdots dx_{k+1}) \\
= \; & \frac{1}{2}E_x((\frac{|b_t - b_{u_k}|^2}{(t-u_k)^2} - \frac{n}{t-u_k})f(b_t)g(b_{u_1}, \cdots, b_{u_k})(\xi)) \\
= \; & \frac{1}{2}E_x(E_x((\frac{|b_t - b_{u_k}|^2}{(t-u_k)^2} - \frac{n}{t-u_k})f(b_t)g(b_{u_1}, \cdots, b_{u_k})(\xi)|\mathcal{F}_{u_k})) \\
= \; & \frac{1}{2}E_x(g(b_{u_1}, \cdots, b_{u_k})(\xi)E_{b_{u_k}}((\frac{|b_{t-u_k} - b_0|^2}{(t-u_k)^2} - \frac{n}{t-u_k})f(b_{t-u_k}))).
\end{aligned}$$

Thus the derivative $\frac{\partial}{\partial t}E_x(f(b_t)g(b_{u_1}, \cdots, b_{u_k})(\xi))$ exists. Moreover

$$\begin{aligned}
& \frac{\partial v_k}{\partial t}(t, x)(\xi) \\
= \; & k!\int_0^t \int_0^{u_{k-1}} \cdots \int_0^{u_2} E_x(f(b_t)g(b_{u_1}, \cdots, b_{u_{k-1}})(\xi)e^{\xi(b_t)})du_1 \cdots du_{k-1} \\
& +\frac{1}{2}k!\int_0^t \int_0^{u_k} \cdots \int_0^{u_2} E_x(f(b_t)g(b_{u_1}, \cdots, b_{u_k})(\xi)(\frac{|b_{t-u_k} - b_0|^2}{(t-u_k)^2}\frac{n}{t-u_k})) \\
& \qquad du_1 \cdots du_k \\
= \; & kE_x(f(b_t)(\int_0^t e^{\xi(b_s)}ds)^{k-1}e^{\xi(b_t)}) + \frac{k}{2}\int_0^t E_x((\int_0^{u_k} e^{\xi(b_s)}ds)^{k-1}\exp[\xi(b_{u_k})]) \\
& \cdot E_{b_{u_k}}((\frac{|b_{t-u_k} - b_0|^2}{(t-u_k)^2} - \frac{n}{t-u_k})f(b_{t-u_k}))du_k.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{k!}\left|\frac{\partial v_k}{\partial t}(t,x)(\xi)\right| \\
\leq\;& Ce^{|\xi|_p}\sum_{k=0}^{\infty}\frac{E_x(\int_0^t e^{|\xi(b_s)|}ds)^k)}{k!}+\frac{C_1}{2}e^{|\xi|_p}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t\frac{E_x(\int_0^{u_k} e^{|\xi(b_s)|}ds)^k)}{\sqrt{t-u_k}}du_k \\
\leq\;& C\exp[|\xi|_p+te^{|\xi|_p}]+C_1\sqrt{t}e^{|\xi|_p}\sum_{k=0}^{\infty}\frac{(te^{|\xi|_p})^k}{k!}<\infty.
\end{aligned}$$

Hence the series $\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial v_k}{\partial t}(t,x)(\xi)$ converges uniformly on any compact subset of $\mathbf{R}^+$ for $x\in\mathbf{R}^n$ and $\xi\in\mathcal{S}_c(\mathbf{R}^n)$. Thus we can interchange the orders to get

$$\frac{\partial F}{\partial t}(t,x)(\xi)=\frac{\partial}{\partial t}\sum_{k=0}^{\infty}\frac{1}{k!}v_k(t,x)(\xi)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial v_k}{\partial t}(t,x)(\xi).$$

Therefore for any fixed $\xi,\eta\in\mathcal{S}_c(\mathbf{R}^n)$, the function $\frac{\partial F}{\partial t}(t,x)(z\xi+\eta), z\in\mathbf{C}$, is entire on $\mathbf{C}$. Furthermore, for all $R\geq 0$,

$$\begin{aligned}
M(R,\xi)\equiv\sup_{|z|=R,z\in\mathbf{C}}&\left|\frac{\partial F}{\partial t}(t,x)(\xi)\right|\leq\sum_{k=0}^{\infty}\frac{1}{k!}\left|\frac{\partial v_k}{\partial t}(t,x)(\xi)\right| \\
\leq\;& C\exp[|\xi|_p+te^{|\xi|_p}]+C_1\sqrt{t}e^{|\xi|_p}\sum_{k=0}^{\infty}\frac{(te^{|\xi|_p})^k}{k!} \\
\leq\;& Ce^{1+te}+C_1\sqrt{t}e^{1+te}=(C+C_1\sqrt{t})e^{1+te},
\end{aligned}$$

for all $\xi\in\mathcal{S}_c(\mathbf{R}^n)$ with $|\xi|_p\leq 1$. Hence $\frac{\partial F}{\partial t}(t,x)(\xi),\xi\in\mathcal{S}_c(\mathbf{R}^n)$, is a quasi-U-functional for $t\geq 0$ and $x\in\mathbf{R}^n$. Thus Lemma 3.3.3 is proved. □

By Lemma 3.3.3 and the characterization in $\mathcal{M}^*$, there exists a unique distribution in $\mathcal{M}^*$, denoted by $\frac{\partial}{\partial t}u(t,x)$, whose S-transform is given by

$$S(\frac{\partial}{\partial t}u(t,x))(\xi)=\frac{\partial F}{\partial t}(t,x)(\xi)$$

for all $t\in\mathbf{R}^+$, $x\in\mathbf{R}^n$, and $\xi\in\mathcal{S}_c(\mathbf{R}^n)$.

THEOREM 3.3.4 *The white noise partial differential equation (3.3.1) has a unique solution in $\mathcal{M}^*$ given by (3.3.6), whose S-transform is given by (3.3.5).*

PROOF. We shall follow the previous notation. For any $\xi\in\mathcal{S}_c(\mathbf{R}^n)$, we show that the series $\sum_{k=0}^{\infty}\frac{1}{k!}\Delta v_k(t,x)(\xi)$ converges uniformly on any compact subset of

$\mathbf{R}^+ \times \mathbf{R}^n$. Let $\varphi(t) = \int_0^t e^{\xi(b_r)}dr$, then the function $\varphi'(t) = e^{\xi(b_t)}$ is continuous since the Brownian motion b_t has continuous paths. Therefore

$$\begin{aligned}
& \lim_{s\to 0} \frac{(\int_s^{s+t} e^{\xi(b_r)}dr)^k - (\int_0^{s+t} e^{\xi(b_r)}dr)^k}{s} \\
= & \lim_{s\to 0} \frac{(\varphi(s+t) - \varphi(s))^k - (\varphi(s+t))^k}{s} \\
= & \lim_{s\to 0} k(\varphi(s+t) - \varphi(s))^{k-1}(\varphi'(s+t) - \varphi'(s)) \\
& - \lim_{s\to 0} k(\varphi(s+t))^{k-1}\varphi'(s+t) \\
= & k(\varphi(t))^{k-1}(e^{\xi(b_t)} - e^{\xi(b_0)}) - k(\varphi(t))^{k-1}e^{\xi(b_t)} \\
= & -k(\varphi(t))^{k-1}e^{\xi(x)}.
\end{aligned}$$

Put $P_t f(x) = E_x(f(b_t))$, then by (3.3.12) and Markov property, we have

$$\begin{aligned}
& \Delta v_k(t,x)(\xi) = \lim_{s\to 0} \frac{P_s v_k(t,x)(\xi) - v_k(t,x)(\xi)}{s} \\
= & \lim_{s\to 0} \frac{E_x(f(b_{t+s})(\int_s^{s+t} e^{\xi(b_r)}dr)^k) - E_x(f(b_t)(\int_0^t e^{\xi(b_r)}dr)^k)}{s} \\
= & \lim_{s\to 0} \frac{E_x(f(b_{t+s})(\int_0^{s+t} e^{\xi(b_r)}dr)^k) - E_x(f(b_t)(\int_0^t e^{\xi(b_r)}dr)^k)}{s} \\
& + \lim_{s\to 0} \frac{E_x(f(b_{t+s})((\int_s^{s+t} e^{\xi(b_r)}dr)^k - (\int_0^{s+t} e^{\xi(b_r)}dr)^k))}{s} \\
= & \frac{\partial v_k}{\partial t}(t,x)(\xi) - kE_x(f(b_t)(\int_0^t e^{\xi(b_r)}dr)^{k-1})e^{\xi(x)}.
\end{aligned}$$

Let $y_m(t,x)(\xi) = \sum_{k=0}^m \frac{1}{k!} v_k(t,x)(\xi)$. Then,

$$\begin{aligned}
& \Delta y_m(t,x)(\xi) = \frac{\partial y_m}{\partial t}(t,x)(\xi) - \sum_{k=0}^m \frac{1}{k!} E_x(f(b_t)(\int_0^t e^{\xi(b_r)}dr)^k)e^{\xi(x)} \\
\to & \frac{\partial F}{\partial t}(t,x)(\xi) - E_x(f(b_t)\sum_{k=0}^\infty \frac{1}{k!}(\int_0^t e^{\xi(b_r)}dr)^k)e^{\xi(x)} \\
= & \frac{\partial F}{\partial t}(t,x)(\xi) - E_x(f(b_t)\exp[\int_0^t e^{\xi(b_r)}dr)]e^{\xi(x)} \\
= & \frac{\partial F}{\partial t}(t,x)(\xi) - F(t,x)(\xi)e^{\xi(x)}.
\end{aligned}$$

Thus the series $\sum_{k=0}^\infty \frac{1}{k!}\Delta v_k(t,x)(\xi)$ converges to $\frac{\partial F}{\partial t}(t,x)(\xi) - F(t,x)(\xi)e^{\xi(x)}$ uniformly on any compact subset of $\mathbf{R}^+ \times \mathbf{R}^n$. Hence we can interchange the orders to get

$$\Delta F(t,x)(\xi) = \Delta(\sum_{k=0}^{\infty} \frac{1}{k!} v_k(t,x)(\xi))$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \Delta v_k(t,x)(\xi) = \frac{\partial F}{\partial t}(t,x)(\xi) - F(t,x)(\xi)e^{\xi(x)}.$$

This implies that $\Delta Su(t,x)(\xi) = \frac{\partial}{\partial t} Su(t,x)(\xi) - Su(t,x)(\xi)e^{\xi(x)}$ for all $\xi \in \mathcal{S}_c(\mathbf{R}^n)$. Thus $\Delta Su(t,x)(\xi)$ is a quasi-U-functional, and there exists a unique distribution in $\mathcal{M}^*$, denoted by $\Delta u(t,x)$, whose S-transform is given by $\Delta Su(t,x)(\xi)$. That is

$$S\Delta u(t,x) = \Delta Su(t,x)(\xi).$$

Summing up we get

$$\frac{\partial}{\partial t} Su(t,x)(\xi) = \Delta Su(t,x)(\xi) + Su(t,x)(\xi)e^{\xi(x)},$$

or

$$S(\frac{\partial u(t,x)}{\partial t})(\xi) = S(\Delta u(t,x) + u(t,x)\diamond :e^{\dot{B}_x}:)(\xi).$$

Since S-transform is injective, we have

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\diamond :e^{\dot{B}_x}: .$$

That is,

$$\frac{\partial u}{\partial t} = \Delta u + :e^{\dot{B}_x}: \diamond u.$$

Furthermore, from (3.3.5), we have $Su(0,x)(\xi) = E_x(f(b_0)) = f(x)$ for all $x \in \mathbf{R}^n$ and all $\xi \in \mathcal{S}_c(\mathbf{R})$. Thus $u(0,x) = f(x)$ for all $x \in \mathbf{R}^n$ by the injectivity of S-transform, and so $u(t,x)$ is a solution of (3.3.1). The uniqueness followes from the uniqueness of the deterministic equation

$$\frac{\partial F}{\partial t}(t,x)(\xi) = \Delta F(t,x)(\xi) + e^{\xi(x)} F(t,x)(\xi).$$

Hence the theorem is proved. □

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Vita

Dongya Zou was born in Hunan, China in 1958. She graduated from Shuangfen First High School, Hunan, China in 1975. She majored in Mathematics at Xiangtan University, Hunan, China and earned a Bachelor of Science degree in 1982 and a Master degree in 1985. She entered the Graduate School at Louisiana State University in 1991 and majored in Mathematics.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Dongya Zou

Major Field: Mathematics

Title of Dissertation: Existence and Uniqueness Theorems for Some White Noise Integral Equations

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

R.A. Litherland

Gisele R Goldstein

Robert Perlis

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